Extended Abstracts

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Densities for Hausdorff measure and rectifiability. Besicovitch's 1/2-conjecture

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Resum *(CAT)*
En aquest treball estudiem un dels conceptes centrals de la teoria geomètrica de la mesura, el de conjunt rectificable, i la seva relació amb les densitats per la mesura de Hausdorff. En aquesta interacció hi ha un dels problemes oberts més antics de la teoria: la conjectura-1/2 de Besicovitch. Estudiem una selecció de resultats rellevants, des dels articles pioners de Besicovitch [1] fins a la millora de Preiss i Tišer [7]. Després, presentem una contribució original: generalitzem a $\mathbb{R}^n$ un exemple donat originalment per Besicovitch en el pla, demostrant-ne les propietats clau i estenent així una cota inferior de la conjectura a dimensió arbitrària.

Keywords: *geometric measure theory, Hausdorff measure, rectifiability, Besicovitch's 1/2-conjecture.*

Abstract

One of the main concepts of geometric measure theory is that of m-rectifiable subsets of $\mathbb{R}^n$, given integers $0 < m \leq n$. They appear as a generalization of the notion of "nice" m-dimensional surfaces, such as C^1 submanifolds, or Lipschitz graphs. They are sets which, up to a set of zero $\mathcal{H}^m$-measure, are contained in a countable union of images of Lipschitz maps with domain in $\mathbb{R}^m$ (where $\mathcal{H}^m$ denotes the m-dimensional Hausdorff measure). For example, for $m = 1$, the 1-rectifiable sets are those which are contained in a countable union of rectifiable curves, again up to a set of zero $\mathcal{H}^1$-measure. On the other side of the coin, we have the purely m-unrectifiable sets, which are those that contain no m-rectifiable subset of positive $\mathcal{H}^m$-measure. One of the goals of geometric measure theory is to characterize rectifiability in terms of other geometric or analytical properties.

To that end, one of the basic tools is that of the densities for the Hausdorff measure. Consider a set $E \subset \mathbb{R}^n$ such that $0 < \mathcal{H}^s(E) < \infty$ for some $0 \leq s \leq n$, which we call an s-set. One defines the upper and lower s-densities of E at a point $x \in \mathbb{R}^n$, denoted as $\Theta^{*s}(E, x)$ and $\Theta^s_*(E, x)$ respectively, as the lim sup and lim inf as $r \to 0$ of

$$\frac{\mathcal{H}^s(E \cap B_r(x))}{(2r)^s}.$$

When both quantities coincide, the limit is called the s-density of E at x.

The densities for the Hausdorff measure and the notion of rectifiability are intimately connected. One of the most important theorems in this direction states that an m-set $E \subset \mathbb{R}^n$ is m-rectifiable if and only if the m-density of E exists and is equal to 1 at $\mathcal{H}^m$-almost all points of E. This is known as the characterization of rectifiability in terms of densities. This line of study was initiated in the pioneering work of Besicovitch [1] in 1938, where he established the result for 1-sets in the plane, i.e., the case $m = 1$ and $n = 2$. It was extended to arbitrary dimension in different stages, with the work of Moore [6], Marstrand [4] and Mattila [5].

Another point of connection between the two topics involves the lower density alone. It was proven by Besicovitch in the same article that if $\Theta^1_*(E,x) > 3/4$ for $\mathcal{H}^1$-almost all points of a 1-set E, then E is automatically 1-rectifiable. Following this idea, we define the following coefficient:

$$\sigma_m(\mathbb{R}^n) := \min\{\sigma > 0 : \text{for any } m\text{-set } E \subset \mathbb{R}^n,\ \Theta^m_*(E,x) > \sigma\,\mathcal{H}^m\text{-a.e. } x \in E \implies E \text{ is } m\text{-rectifiable}\}.$$

The previously stated result of Besicovitch translates to the bound $\sigma_1(\mathbb{R}^2) \leq 3/4$. Moreover, in the same article in 1938 he provided an example of a purely 1-unrectifiable set P which satisfies $\Theta^1_*(P,x) = 1/2$ at $\mathcal{H}^1$-almost all $x \in P$; a formal proof of this fact appeared later in a paper by Dickinson [3] in 1939. This way, they proved the lower bound $\sigma_1(\mathbb{R}^2) \geq \frac{1}{2}$. With this in mind, Besicovitch conjectured that the exact value of $\sigma_1(\mathbb{R}^2)$ is $1/2$, which is now known as *Besicovitch's* $1/2$*-conjecture*.

Further improvements to this bound have been obtained since then. In 1992, Preiss and Tišer [7] refined the estimate to $\sigma_1(\mathbb{R}^n) \leq (2+\sqrt{46})/12 < 59/80$, which holds for all $n \geq 2$ (for all metric spaces, in fact). Recently, in 2024, Camillo De Lellis et al. [2] established that $\sigma_1(\mathbb{R}^n) \leq 7/10$, which is currently the best known upper bound.

In higher dimensions (for $m > 1$), no good upper bounds are known for $\sigma_m(\mathbb{R}^n)$. On the other hand, the same lower bound remains valid; in this work, we generalize Besicovitch's example in the plane to arbitrary dimensions, thereby showing

$$\sigma_m(\mathbb{R}^n) \geq \frac{1}{2}, \quad \text{for any } 0 < m < n.$$

This is an original contribution from this work.

References

[1] A.S. Besicovitch, On the fundamental geometrical properties of linearly measurable plane sets of points (II), *Math. Ann.* **115(1)** (1938), 296–329.

[2] C. De Lellis, F. Glaudo, A. Massaccesi, D. Vittone, Besicovitch's 1/2 problem and linear programming, Preprint (2024). `arXiv:2404.17536`.

[3] D.R. Dickinson, Study of extreme cases with respect to the densities of irregular linearly measurable plane sets of points, *Math. Ann.* **116(1)** (1939), 358–373.

[4] J.M. Marstrand, Hausdorff two-dimensional measure in 3-space, *Proc. London Math. Soc. (3)* **11** (1961), 91–108.

[5] P. Mattila, Hausdorff m regular and rectifiable sets in n-space, *Trans. Amer. Math. Soc.* **205** (1975), 263–274.

[6] E.F. Moore, Density ratios and $(\phi, 1)$ rectifiability in n-space, *Trans. Amer. Math. Soc.* **69** (1950), 324–334.

[7] D. Preiss, J. Tišer, On Besicovitch's $\frac{1}{2}$-problem, *J. London Math. Soc. (2)* **45(2)** (1992), 279–287.