Extended Abstracts

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Density of hyperbolicity in families of complex rational maps

Francesc Timoner Vaquer
Universitat de Barcelona
francesc.timoner-vaquer@utoulouse.fr

Resum *(CAT)*
Un dels problemes oberts centrals és la densitat d'hiperbolicitat. En aquest treball ho investiguem en la dinàmica complexa unidimensional, i ens concentrem en el cas polinòmic (cas particular d'una funció racional) com a model on els mecanismes principals poden ser exposats i comprovats en detall. La via procedimental és clara: primer, la construcció de peces de puzle en un entorn del conjunt de Julia; segon, l'ús d'aquestes per definir una funció de caixa complexa; i finalment, l'aplicació de teoremes de rigidesa a aquestes. Aquest procés tradueix la informació combinatòria en rigidesa per als polinomis, demostrant que un polinomi no renormalitzable pot ser aproximat per un polinomi hiperbòlic.

Keywords: *complex dynamics, holomorphic dynamics, rational maps, hyperbolicity, renormalisation, complex box mapping.*

Societat Catalana de Matemàtiques
Institut d'Estudis Catalans

Abstract

A *rational map* is a holomorphic analytic function $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ on the Riemann sphere that can be written as the quotient of two coprime polynomials; equivalently, $f(z) = \frac{P(z)}{Q(z)}$, where P, Q are complex polynomials of some degree. The degree of f is defined as $d = \max(\deg P, \deg Q)$, and we assume that $d \geq 2$. In the particular case where Q is a constant, f is just a polynomial. Rational maps of degree $d \geq 2$ form a finite-dimensional space, so exploring this parameter space is feasible. Every rational map of degree $d \geq 2$ has $2d-2$ critical points (counting multiplicity), and near these points, the map behaves like $z \mapsto z^k$, so it is highly contracting and fails to be injective. Away from the critical points, f is a local homeomorphism.

Definition (Hyperbolic rational map)**.** A rational map is said to be *hyperbolic* if all its critical points are in the basins of attracting periodic points.

Conjecture (Density of hyperbolicity)**.** *The hyperbolic rational maps form an open and dense set in the space of all rational maps of a given degree.*

Definition. We say that a map is *non-renormalisable* if it does not admit any polynomial-like restriction for any iteration with connected filled-in Julia set.

The main goal of this thesis is to deconstruct, understand all details and prove the following theorem:

Main Theorem (Theorem 1.3 in [3])**.** *Let f be a non-renormalisable polynomial of degree $d \geq 2$, without neutral periodic points. Then, f can be approximated by a sequence of hyperbolic polynomials (g_i) of the same degree.*

To tackle the problem we review and combine several fundamental tools: puzzle piece decompositions so we can consider returns and track symbolically critical orbits, Böttcher coordinates near infinity that linearize escaping behaviour, holomorphic motions to follow dynamical objects across parameters, and quasi-conformal conjugacies to transfer geometric control between maps. Also, we suppose our map is non-renormalisable: for a rational map, one demands that its critical orbits do not return in small neighbourhoods in a "periodic way". These maps are often rigid, in the sense that their combinatorial structure determines their geometry. Finally, and most importantly, we make use of complex box mappings as an induced map defined on a disjoint union of topological discs that captures return dynamics of critical orbits inside a controllable domain ("upgrade" of the famous polynomial-like maps). These are flexible enough to encode both local renormalisation behaviour and global combinatorial constraints.

The seminal paper [3] (our main reference) lacks explicit technical assumptions in its statements and proofs. This makes some of their statements, as written in that paper, not entirely correct. Some assumptions were implicit or not considered, for example, the dynamically natural property of complex box mappings. Some parts, claimed to be straightforward, are not. In [1] they clarified and fixed some results on rigidity of polynomials and box mappings, but the theorem stated above remains unclear. So for the first time in the literature of complex dynamics, we provide detailed explanations for each part of the proof of that theorem. We consider the implications and ensure validity, especially when considering the dynamically natural property of box mappings. Our aim is to review the existing literature ([4, 2]), emphasising crucial aspects, and comprehensively understand the tools required for the theorem's proof. We aim to encapsulate them in a "black-box" and use them to advance research, for instance, to establish the density of hyperbolicity in other families of rational maps. We believe this meticulous deconstruction and attention to detail can significantly contribute to the general public's understanding of the subject matter.

The proof of the Main Theorem lies on a construction of dynamically natural box mappings for non-renormalisable polynomials without neutral periodic points together with a verification of the hypotheses needed to invoke rigidity theorems. In rigid families, topologically conjugate maps are automatically more regular (e.g., quasi-conformal or conformal in complex dynamics). By another of the main theorems needed (Theorem 6.1 in [1]), combinatorially equivalent non-renormalisable dynamically natural complex box mappings are rigid, and hence, quasi-conformally conjugate. This result, along with other known or basic notions, leads to the quasi-conformal rigidity of non-renormalisable polynomials. Consequently, the original polynomial is approximated, in the uniform topology on compact sets, by hyperbolic polynomials; hence density of hyperbolicity holds for the class considered.

References

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