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Unique preduals and free objects in Banach spaces

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Resum *(CAT)*
Estudiem quan un espai de Banach té un únic predual, centrant-nos primer en les funcions holomorfes acotades al disc unitat i analitzant la demostració d'Ando. Considerem com estendre el resultat a diverses variables, on apareixen dificultats tècniques. També tractem diferents condicions suficients per garantir la unicitat i el cas de reticles de Banach.

Keywords: *unique predual, Banach space, bounded holomorphic functions, Property (X), L-embedded space, free Banach lattice.*

Abstract

A *predual* of a Banach space X is a Banach space Y such that there exists an isomorphism $Y^* \to X$. When X admits only one such space Y up to isometric isomorphism, we say that X has a *unique predual*. The problem of determining when a Banach space has a unique predual is a central one in functional analysis, starting in the works of Dixmier (1948) and Ng [5] and studied by Sakai, Ando, Godefroy, Pfitzner, and others. Classical examples of spaces with unique preduals include von Neumann algebras by Sakai (1971), the space of bounded holomorphic functions on the complex disk $H^\infty(\mathbb{D})$ by Ando [1], and separable L-embedded Banach spaces by Pfitzner [6]. Godefroy's survey [3] remains a key reference summarizing these developments and listing open problems.

This project revisits the uniqueness problem with emphasis on the space of bounded holomorphic functions, $H^\infty(U)$, defined on an open subset $U \subset \mathbb{C}^n$. We review Ando's original proof of the uniqueness of the predual of $H^\infty(\mathbb{D})$, which identifies the space $L^1(\mathbb{T})/H^1_0(\mathbb{T})$ as its unique predual. We present a detailed proof following both Ando's original formulation [1] and a later one from Garnett [2], filling several gaps left unproved in the literature.

We extend Ando's result to the case where U is a disjoint union of simply connected open subsets of the complex plane. Using Mujica's notion of the holomorphic free Banach space $G^\infty(U)$ given in [4], characterized by the universal property

$$H^\infty(U, F) \cong L(G^\infty(U), F),$$

we explicitly construct an isometric isomorphism

$$G^\infty(U) \cong \bigoplus_{\alpha \in A} G^\infty(U_\alpha),$$

where $U = \bigsqcup_{\alpha \in A} U_\alpha$. This allows us to prove that $H^\infty(U)$ has a unique predual whenever U is such a disjoint union, thus generalizing Ando's result.

We then attempt to extend the result to several complex variables, considering $H^\infty(\mathbb{T}^n)$ and $H^\infty(B_n)$. Following a different path of proving uniqueness in the one-dimensional case, we reduce the problem of proving that $H^\infty(B^n)$ has a unique predual to showing that $B_{H^1_0(\mathbb{S}_n)}$ is $\|\cdot\|_p$-closed for some $p \in (0,1)$. However, this higher-dimensional setting presents several challenges. When passing from one variable to several, the space $H^1_0(\mathbb{S}_n)$ is no longer contained in the Hardy space $H^1(\mathbb{B}_n)$, so pre-compactness arguments from Hardy space theory cannot be applied. Consequently, no conclusive results were obtained in this direction.

This work also reviews techniques guaranteeing uniqueness of preduals in broader settings. Two sufficient conditions are revisited: *Property (X)* (Godefroy–Talagrand, 1980), which ensures that X is the unique predual of X^*, and the notion of *L-embedded spaces*, for which separable cases were solved by Pfitzner [6]. Finally, we explore the Banach lattice setting, introducing the *free Banach lattice* FBL[E] (Avilés–Rodríguez–Tradacete, 2015). Although we explore possible definitions for predual equivalence in this setting, we find difficulties, particularly because the space of lattice homomorphisms is not a vector space, thus leaving this problem for future research.

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