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Idempotent elements of the group algebra

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L'objectiu d'aquest treball és estudiar els elements idempotents centralment primitius de l'àlgebra de grup i desenvolupar un mètode per al seu càlcul en el cas de cossos finits. A partir de la teoria de representacions de grups finits i de resultats sobre mòduls, àlgebres i extensions de cossos, s'introdueix el concepte de cos d'escissió per a un grup. Finalment, s'explora com l'acció de Galois sobre l'àlgebra de grup definida sobre aquests cossos permet obtindre aquests idempotents del cos original.

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Abstract

This work focuses on the study of idempotent elements of group algebras, with particular emphasis on centrally primitive idempotents. These elements are fundamental because they allow the algebra to be decomposed into simpler blocks. The importance of centrally primitive idempotents lies in the fact that each of them generates one of these blocks and, moreover, they form a basis for the centre of the algebra, which completely defines its structure.

The main objective is to develop an explicit and practical method for calculating these idempotents over fields whose characteristic does not divide the order of the group (which we will assume to be finite), and which are often not algebraically closed. This is no easy task, since many results in representation theory rely on the latter property (see [3]) and are not valid in a more general context. For this reason, we resort to the concept of a splitting field for a group (see [1, 2]), which generalises the algebraically closed field, providing a theoretical framework that guarantees the validity of many classical results, including the expression of these idempotents.

The method we develop, often known as Galois descent, consists of exploiting the expression of centrally primitive idempotents of the group algebra over a splitting field. The idea is to consider a finite Galois extension of the original field that is a splitting field for the group; in this extension, the Galois group acts on these idempotents. The expression of these idempotents is known since they are defined over a splitting field, and it can be shown that the sum of the orbits resulting from this action ultimately gives us the centrally primitive idempotents we are looking for in the original field (see [4]). This method is significantly simpler than other approaches, such as the one in [5], which relies on the computation of a division ring's dimension—a generally non-trivial task. Furthermore, we prove that both methods are equivalent by summing over a general orbit to obtain the expression given in [5].

To illustrate the procedure, we conclude with a detailed application to finite fields, where the efficiency of our approach becomes particularly evident. The practicality of the method lies not only in the simplicity of the orbit computations—thanks to the cyclic nature of the Galois group generated by the Frobenius automorphism—but also in the theoretical results previously developed in this work, which directly provide the corresponding splitting fields. In this example, we first establish the identification between characters over the splitting field and ordinary characters via Brauer characters (see again [3]), and then carry out the explicit computation of the idempotents, thereby demonstrating the applicability and strength of our self-contained approach.

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