Fusion theorems and applications

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Resum *(CAT)*
En teoria de grups finits, molts resultats clàssics impliquen subgrups de Sylow. Una direcció natural és generalitzar-los mitjançant subgrups de Hall. En aquest treball, mostrem com un resultat de Wielandt permet fer-ho eficaçment. Presentem dues aplicacions: una relacionada amb el teorema de fusió d'Alperin, i una altra amb el subnormalitzador, un concepte menys conegut però amb connexions recents amb la teoria de caràcters.

Keywords: *fusion in groups, subnormalizer.*

Abstract

In finite group theory, many results are formulated in terms of Sylow subgroups and rely heavily on the classical Sylow theorems. These results are central to the local-global philosophy of the subject, where local properties of subgroups provide valuable information about the structure of the whole group.

Whenever such theorems are established, a natural line of inquiry arises: can these results be generalized beyond Sylow subgroups? One promising direction involves replacing Sylow subgroups with Hall subgroups, which are more general but retain many desirable properties when they exist. However, such generalizations often require more sophisticated tools, since the theory of Hall subgroups is not as robust or widely applicable as Sylow theory in general finite groups.

In this work, we focus on a classical but perhaps underappreciated result by Wielandt, which proves to be a powerful instrument in extending certain Sylow-based statements to more general contexts involving Hall subgroups. Wielandt's theorem offers a unifying perspective that opens the door to new applications.

We present two main applications of this approach. The first concerns Alperin's fusion theorem, first proved by Alperin in [1], a fundamental result describing how conjugacy in a Sylow p-subgroup is controlled in terms of the local structure. This theorem is important in some conjectures in representation theory and character theory. We will show how Wielandt's result can be used to extend aspects of this theorem beyond the Sylow subgroups, providing a more flexible framework for studying fusion phenomena.

The second application involves a less well-known concept: the subnormalizer of a subgroup. This notion, mainly studied by Carlo Casolo in [2], tries to generalize the concept of normalizer. Subnormalizers offer an alternative lens through which one can examine the internal structure of a finite group. Recent developments

show that this concept is not merely technical: it is connected to new conjectures in character theory and may lead to fresh insights into the interplay between subgroup structure and representation theory.

Both applications illustrate how classical tools, when viewed from a modern perspective, can be effectively repurposed to approach contemporary problems in group theory. The ideas we present highlight the ongoing relevance of results like Wielandt's theorem and demonstrate the value of re-examining classical results through new conceptual frameworks.

Acknowledgements

References

[1] J.L. Alperin, Sylow intersections and fusion, *J. Algebra* **6** (1967), 222–241.

[2] C. Casolo, Subnormalizers in finite groups, *Comm. Algebra* **18(11)** (1990), 3791–3818.