Extended Abstracts

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Atypical values of complex polynomial functions

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Resum *(CAT)*
Des de 1983, amb el treball de Broughton, s'han introduït diverses condicions de regularitat a l'infinit per a un polinomi complex f que garanteixen l'absència de valors crítics a l'infinit, és a dir, de valors atípics de f que no són valors crítics. En aquest treball recollim les condicions de regularitat més rellevants i estudiem les relacions que hi ha entre elles. En particular, responem a dues preguntes obertes proposades per Dũng Tráng Lê i J.J. Nuño-Ballesteros a [3].

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Abstract

The topology of complex polynomial functions $f\colon \mathbb{C}^n \longrightarrow \mathbb{C}$ has been object of considerable study in recent decades. In particular, a central goal is to understand how the topology of the fibers $f^{-1}(c)$, $c \in \mathbb{C}$, changes. In this context, the concept of *locally trivial fibrations* plays a key role. Specifically, if f is locally a trivial fibration at $c \in \mathbb{C}$, then the topology of the fibers near c remains unchanged. The points $c \in \mathbb{C}$ where f fails to be locally a trivial fibration are called atypical values of f. The set of all atypical values of f is denoted by $\operatorname{Atyp} f$. In [4], Thom proved the finiteness of the set of atypical values. However, determining precisely this set is a major open problem.

Among the atypical values, one has the critical values, *i.e.*, $f(\Sigma f) \subset \operatorname{Atyp} f$, where Σf is the set of points $x \in \mathbb{C}^n$ where $\mathrm{d}f_x = 0$. In general, this inclusion is strict. Over the past decades, several *regularity conditions at infinity* for f have been introduced in order to guarantee the equality $f(\Sigma f) = \operatorname{Atyp} f$.

The first one is the notion of *tameness*, which was introduced by Broughton in [1] and [2]. In [5], Tibăr compiles some other regularity conditions at infinity, such as the *Malgrange Condition* (which generalizes the notion of tameness) and the *ρ-regularity at infinity*, where ρ is a *control function*. The following chain of implications is well-known:

$$\begin{aligned} f \text{ is tame} &\Longrightarrow f^{-1}(c) \text{ satisfies the Malgrange Condition} \\ &\Longrightarrow f^{-1}(c) \text{ is } \rho_E\text{-regular at infinity.} \end{aligned}$$

Most recently, in [3], Dũng Tráng Lê and J.J. Nuño Ballesteros introduced the notion of *atypical values from infinity*. In this paper, they generalize the Broughton's Global Bouquet Theorem in [2]. The paper concludes by posing several open questions aimed at gaining a deeper understanding of atypical values from infinity. Namely,

1. Is it true that, if f is tame, then f does not have atypical values from infinity?
2. Does a fiber $f^{-1}(c)$ which satisfies the Malgrange Condition correspond to a value c which is not an atypical value from infinity?

In this work we review all these definitions and explain our main contribution:

$$f^{-1}(c) \text{ is } \rho\text{-regular at infinity} \Longrightarrow c \text{ is not an atypical value from infinity.}$$

Using this result, we obtain an extension of the previous chain of implications:

$$\begin{aligned} f \text{ is tame} &\Longrightarrow f^{-1}(c) \text{ satisfies the Malgrange Condition} \\ &\Longrightarrow f^{-1}(c) \text{ is } \rho_E\text{-regular at infinity} \\ &\Longrightarrow c \text{ is not an atypical value from infinity.} \end{aligned}$$

This answers the first question of the authors in [3] and gives the right implication for the second one. The other implication remains open.

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