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Use of Fourier series in $\mathbb{S}^2$ to approximate star-shaped surfaces

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Resum *(CAT)*
Primer ens centrem a estendre la noció de sèrie de Fourier, estudiant com podem representar funcions de quadrat integrable sobre varietats de Riemann. Per fer això ens ajudem del teorema de Hodge, que ens permetrà trobar bases d'aquests espais a partir del laplacià.
Després veiem com aquest mètode es pot utilitzar per trobar les sèries de Fourier per a funcions periòdiques, i per a les funcions $L^2(\mathbb{S}^2)$, el cas principal. També discutim com estimar l'error en norma L^2, i implementem totes les fórmules que es troben a l'article a un programa per poder visualitzar els resultats obtinguts.

Abstract *(ENG)*
First we will focus in extending the notion of Fourier series, studying how can we represent functions of integrable square over Riemannian manifolds. To do this we will use the Hodge theorem, that will allow us to find basis of these spaces through the Laplacian.
Then we will see how this method can be used to find the Fourier series for periodic functions, and for the functions $L^2(\mathbb{S}^2)$, our main case of study. We will also discuss how to estimate the L^2-error, and we implement all the formulas found in the article in a program to be able to visualize the obtained results.

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1. Introduction

This article will follow the steps that were set in my bachelor's thesis and explain how to build Fourier series to be able to represent star-shaped surfaces.

In particular we will look at the construction of the Laplacian operator Δ induced by a metric g, on a set of functions $L^2(X)$, where X is a Riemannian manifold, and how we can use the Hodge theorem to look for basis of the function space $\{\psi_i\}_{i\in\Lambda} \in L^2(X)$ in the eigenvectors of the Laplacian $\Delta\psi_i = \lambda_i\psi_i$.

We will primarily focus our attention on the case for $\mathbb{S}^2$, where such basis will be given by the spherical harmonics, and we will study how to apply these formulas for computational use, and also how to a priori estimate the L^2-error for a certain amount of coefficients.

All of this culminating in a desktop application, that given a star-shaped surface triangulation will use the formulas explained on the article to represent that shape as a Fourier series, proving the applicability of the mathematics explored on my thesis.

2. Construction of the Laplacian on a Riemannian manifold

Given (X, g) a compact Riemannian manifold, for further calculations we will assume

$$\begin{aligned} \exists\Phi\colon U \subset \mathbb{R}^n &\longrightarrow X \\ (x_1, \dots, x_n) &\longrightarrow \Phi(x_1, \dots, x_n) \end{aligned}$$

a parameterization with a dense image, $\overline{\Phi(U)} = X$.

Definition 2.1. Given the prior manifold we define the Hilbert space

$$L^2(X) = \{f : X \longrightarrow \mathbb{R},\ f \text{ integrable square}\}$$

with the scalar product for $f, h \in C^\infty(X)$,

$$\langle f, h\rangle = \int_X fh\, dV_g,$$

where dV_g is the differential induced by the metric.

To build the Laplacian we will work with the dense subset $C^\infty(X) \subset L^2(X)$.

Definition 2.2. Let $\Omega^0(X) = C^\infty(X)$ be the space of differential 0-forms of X. We define

$$\Omega^1(X) = \{\omega = f_1\, dx_1 + \cdots + f_n\, dx_n \mid f_i \in \Omega^0(X)\}$$

the space of differential 1-forms of X.

Definition 2.3. We define the differential operator d as

$$d\colon \Omega^0(X) \longrightarrow \Omega^1(X)$$
$$f \longrightarrow \frac{\partial f}{\partial x_1}\,dx_1 + \cdots + \frac{\partial f}{\partial x_n}\,dx_n.$$

We intend to use the differential operator to build the Laplacian as a self-adjoint operator, $\Delta = d^* \circ d$. Therefore we need to also build a Hilbert space with a well defined scalar product on $\Omega^1(X)$.

Definition 2.4. Let $L^2(X) = A^0(X)$, then we define

$$A^1(X) = \{\omega = f_1\,dx_1 + \cdots + f_n\,dx_n \mid f_i \in A^0(X)\}$$

the set of L^2 1-forms of X, and we see $\overline{\Omega^1(X)} = A^1(X)$.

Now we want to build the tensor $g^*\colon T^*X \oplus T^*X \to \mathbb{R}$.

Let $e_1, \dots, e_n \in T_xX$ be an orthonormal basis for g, then the dual basis of the 1-forms $\omega_1, \dots, \omega_n \in T_x^*X$ so that $\omega_i(e_j) = \delta_{ij}$ will be an orthonormal basis for g^* of T^*X. Through this basis g^* is well defined.

Definition 2.5. With what we have seen so far we can introduce then the scalar product in $A^1(X)$ with this tensor. Given $\omega, \eta \in \Omega^1(X)$,

$$\langle \omega, \eta \rangle_{\Omega^1} = \int_X g_x^*(\omega(x), \eta(x))\,dV_g.$$

This scalar product satisfies the properties of a Hilbert space. Therefore, we have another Hilbert space in $A^1(X)$.

Now we can use the definition of the adjoint operator to find the adjoint of the differential, and finally build the Laplacian operator.

$$\langle df, \omega \rangle_{A^1} = \langle f, d^*\omega \rangle_{A^0}, \quad \forall f \in \Omega^0(X),\ \forall \omega \in \Omega^1(X),$$
$$\Delta = d^* \circ d.$$

3. The Hodge theorem

Now that we have a self-adjoint operator inside a Hilbert space, we will be using the Hodge theorem, specific for this case, to back up our claim that we can find basis for this function space using the Laplacian.

In this article we will just announce the theorem without further proof. For a proper proof as well as more in depth information please refer to [5, p. 32], where the Hodge theorem is established and proven.

Theorem 3.1. *Let (M,g) be a compact Riemannian manifold, oriented and connected. There exists an orthonormal basis for $L^2(M,g)$ that consists of eigenvectors of the Laplacian. All the eigenvalues are real and positive, except for 0 which is an eigenvalue of multiplicity 1. Every eigenvalue has finite multiplicity, and they only accumulate at infinity.*

Once we find those eigenvectors $\{\psi_i\}_{i\in\Lambda} \in L^2(X)$ that form a basis, we can write any given element $f \in L^2(X)$ as its Fourier series

$$f = \sum_{i\in\Lambda} \alpha_i \psi_i, \tag{1}$$

where the coefficients are the ones obtained through the scalar product

$$\alpha_i = \langle f, \psi_i \rangle. \tag{2}$$

4. Construction of the common Fourier series

To begin with the examples let us focus on the case for $X = \mathbb{S}^1$, with the parameterization

$$\begin{aligned}\Phi\colon (0, 2\pi) &\longrightarrow \mathbb{S}^1 \subset \mathbb{R}^2\\ x &\longrightarrow (\cos x, \sin x).\end{aligned}$$

We will be using the metric $g = \frac{dx \otimes dx}{4\pi^2}$, giving us the following scalar product, given $f, h \in A^0(\mathbb{S}^1)$,

$$\langle f, h \rangle_{A^0} = \frac{1}{2\pi} \int_0^{2\pi} f(x)h(x)\, dx.$$

And following the definition for g^* we obtain the scalar product for $A^1(\mathbb{S}^1)$. Given $\omega, \eta \in A^1(\mathbb{S}^1)$, with $\omega(x) = f(x)\, dx$ and $\eta(x) = h(x)\, dx$,

$$\langle \omega, \eta \rangle_{A^1} = \frac{1}{2\pi} \int_0^{2\pi} g_x^*(f(x)\, dx, h(x)\, dx)\, dx = 2\pi \int_0^{2\pi} f(x)h(x)\, dx.$$

Then applying the definition of the adjoint operator we obtain, given any $f \in \Omega^0(\mathbb{S}^1)$ and any $\omega \in \Omega^1(\mathbb{S}^1)$ with $\omega = h(x)\, dx$,

$$\langle df, \omega \rangle_{A^1} = \langle f, d^*\omega \rangle_{A^0},$$

$$2\pi \int_0^{2\pi} f'(x)h(x)\, dx = \frac{1}{2\pi} \int_0^{2\pi} f(x)(d^*\omega)(x)\, dx.$$

Integrating by parts and after some algebra we end up obtaining the expression for the Laplacian

$$\Delta = -4\pi^2 \partial_x^2.$$

Now we want to look for the eigenvectors of this operator. Therefore we want to obtain a set of orthonormal functions $\{\psi_i\}_{i\in\Lambda} \in L^2(\mathbb{S}^1)$ that satisfy the equation $\Delta\psi_i = \lambda_i\psi_i$. If we develop the prior expression, we see

$$\begin{aligned}\Delta\psi &= \lambda\psi,\\ -4\pi^2\partial_x^2\psi(x) &= \lambda\psi(x),\\ 4\pi^2\psi'' &= -\lambda\psi,\end{aligned}$$

giving us the very natural orthonormal eigenvectors

$$\begin{aligned}\psi_0(x) &= 1,\\ \psi_n(x) &= \sqrt{2}\cos(nx),\\ \phi_n(x) &= \sqrt{2}\sin(nx).\end{aligned}$$

As we can easily prove by solving the differential equation, these are all the solutions despite lineal combinations of them. Therefore, now as described in the prior section we can create the Fourier series using this basis, giving us the common example for real Fourier series we all know. Given any $f \in L^2(\mathbb{S}^1)$, we can write it as

$$
\begin{aligned}
f(x) &= A_0 + \sum_{n=1}^{\infty}(A_n \cos(nx) + B_n \sin(nx)), \quad x \in (0, 2\pi), \\
A_0 &= \frac{1}{2\pi}\int_0^{2\pi} f(x)\, dx, \\
A_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\, dx, \\
B_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\, dx.
\end{aligned}
$$

We can also consider the case for $L^2(\mathbb{S}^1, \mathbb{C})$. In this space almost all the process is exactly the same, and we obtain the same self-adjoint operator. The only difference is that when defining the scalar product, this one requires a symmetry by the conjugate, therefore we define given $f, h \in L^2(\mathbb{S}^1, \mathbb{C})$,

$$
\langle f, h\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(x)\overline{h(x)}\, dx.
$$

For this particular example $C^\infty(\mathbb{S}^1, \mathbb{C})$, the most common basis will be

$$
\psi_n(x) = e^{inx}, \quad \forall n \in \mathbb{Z} \text{ with } \lambda_n = 4\pi^2 n^2.
$$

Finally, given any $f \in L^2(\mathbb{S}^1, \mathbb{C})$, we can write it as

$$
\begin{aligned}
f(x) &= \sum_{n=-\infty}^{\infty} \alpha_n e^{inx}, \\
\alpha_n &= \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}\, dx.
\end{aligned}
$$

5. Case for $L^2(\mathbb{S}^2)$, the spherical harmonics

To start with this case we will also start defining the parameterization,

$$
\begin{aligned}
\Phi \colon (0, 2\pi) \times (0, \pi) &\longrightarrow \mathbb{S}^2 \subset \mathbb{R}^3 \\
(\varphi, \theta) &\longrightarrow (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta).
\end{aligned}
$$

Through this parameterization we can find a Riemannian metric induced from $\mathbb{R}^3$, giving us

$$
g = \sin^2\theta\, d\varphi \otimes d\varphi + d\theta \otimes d\theta.
$$

Therefore the scalar product will be

$$
\langle f, h\rangle_{A^0} = \int_{\mathbb{S}^2} fh\, dV_g = \int_0^{\pi}\int_0^{2\pi} f(\varphi, \theta)h(\varphi, \theta)\sin\theta\, d\varphi\, d\theta.
$$

And following the definition for g^* we obtain the scalar product for $A^1(\mathbb{S}^2)$, given

$$\omega(\varphi,\theta) = f_1(\varphi,\theta)\,d\varphi + f_2(\varphi,\theta)\,d\theta,$$
$$\eta(\varphi,\theta) = h_1(\varphi,\theta)\,d\varphi + h_2(\varphi,\theta)\,d\theta,$$

we obtain

$$\langle \omega,\eta\rangle_{A^1} = \int_{\mathbb{S}^2} g^*(\omega,\eta)\,dV_g = \iint \left(\frac{f_1 h_1}{\sin\theta} + f_2 h_2 \sin\theta\right)\,d\varphi\,d\theta.$$

Applying the definition of the adjoint operator we obtain, given $f\in\Omega^0(\mathbb{S}^2)$ and $\omega\in\Omega^1(\mathbb{S}^2)$,

$$\langle df,\omega\rangle_{A^1} = \langle f, d^*\omega\rangle_{A^0}.$$

After some calculations we end up obtaining the following expression for the Laplacian,

$$\Delta = -\left(\frac{\partial_\varphi^2}{\sin^2\theta} + \frac{\partial_\theta(\sin\theta\,\partial_\theta)}{\sin\theta}\right).$$

At this point we introduce the spherical harmonics, which are functions with the following expression.

Definition 5.1. Let $\ell\in\mathbb{N}\cup\{0\}$ and $m\in\mathbb{Z}$ with $|m|\le\ell$. We define the function $Y_\ell^m\colon\mathbb{S}^2\to\mathbb{R}$ parameterized with the coordinates we have been using so far as

$$Y_\ell^m(\varphi,\theta) = \begin{cases} \sqrt{\dfrac{(2\ell+1)(\ell-m)!}{2\pi(\ell+m)!}}P_\ell^m(\cos\theta)\cos(m\varphi) & \text{if } m>0,\\ \sqrt{\dfrac{2\ell+1}{4\pi}}P_\ell^0(\cos\theta) & \text{if } m=0,\\ \sqrt{\dfrac{(2\ell+1)(\ell-|m|)!}{2\pi(\ell+|m|)!}}P_\ell^{|m|}(\cos\theta)\sin(|m|\varphi) & \text{if } m<0,\end{cases} \tag{3}$$

where P_ℓ^m are the associated Legendre polynomials. They are defined as the canonical solutions of the general Legendre equation

$$(1-x^2)\frac{d^2}{dx^2}P_\ell^m(x) - 2x\frac{d}{dx}P_\ell^m(x) + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right]P_\ell^m(x) = 0. \tag{4}$$

These functions are the orthonormal basis obtained by the eigenvectors of the Laplacian. Unfortunately given the length constrains of this article the proper proof of such claim is not possible. For the interested reader I strongly recommend checking the full thesis for a very interesting proof as well as a more in depth explanation of all the steps we have done before and all the results we have seen.

Nevertheless I will try to give a somewhat satisfying overview of the proof found in the thesis. First we start by proving that the spherical harmonics are orthonormal; for this and further steps ahead we use [4, Chapter 14], [1, pp. 331–341], where a lot of the properties of the Legendre polynomials are found as well as some recurrences useful for their computation. In this case we use the Legendre polynomials orthogonality,

$$\int_{-1}^{1} P_\ell^m(x)P_k^m(x)\,dx = \frac{2(\ell+m)!}{(2\ell+1)(\ell-m)!}\delta_{\ell k}.$$

Which finally ends up helping us find the orthonormality result

$$\langle Y_\ell^m, Y_k^n\rangle = \delta_{\ell k}\delta_{mn}.$$

Then we evaluate the Laplacian of the spherical harmonics. To simplify the expression we use the substitution $v = \cos\theta$, where $v' = -\sin\theta = -\sqrt{1-v^2}$; with some rearrangement we can use (4) to get rid of one of the derivatives, allowing us to simplify the expression and with some algebra we obtain $\Delta Y_\ell^m = \ell(\ell+1)Y_\ell^m$, which proves that the spherical harmonics are eigenvectors of the Laplacian with eigenvalues $\lambda_\ell = \ell(\ell+1)$.

Finally we have to prove that these eigenvalues are sufficient to make a basis for $L^2(\mathbb{S}^2)$. For this final step we inspire on the beautiful proof found in Chapter 7 of the book *Groupes et symétries* [2]. We begin with the Stone–Weierstrass theorem.

Theorem 5.2. *Let X be a compact Hausdorff space and A a sub-algebra of the space of continuous functions from X to the real numbers $C^0(X)$, that has a constant non-zero function. Then A is dense in $C^0(X)$ if and only if it separates points.*

This tells us that the polynomials are dense for $C^0(X)$ as long as X is a compact Hausdorff space. Therefore the proof centers itself in proving that the harmonic polynomials are a basis for the restriction to $\mathbb{S}^2$ of all polynomials in $\mathbb{R}^3$, more specifically there are a total of $2\ell+1$ homogeneous harmonic polynomials of degree ℓ, and all together they form a basis of the polynomials in $\mathbb{R}^3$ restricted to the sphere. And therefore applying the Stone–Weierstrass theorem they are a basis for all continuous functions restricted to the sphere.

The second part of the proof uses the results we obtained earlier from the spherical harmonics to prove that these are indeed the restriction to $\mathbb{S}^2$ of homogeneous harmonic polynomials of degree ℓ, and since we have $2\ell+1$ spherical harmonics for every ℓ value, we conclude that these are a basis for $L^2(\mathbb{S}^2)$.

6. A priori estimations of the L^2-error

It would be interesting for further applications of the formulas we have seen so far if we had an a priori estimate of how many coefficients we need to compute to obtain a desired relative L^2-error. For a subset of frequencies $L \subset \Lambda$ we define this error as

$$\epsilon = \frac{||f - f^L||}{||f||},$$

where $f^L = \sum_{i\in L}\langle f, \psi_i\rangle\psi_i$.

To do this we will be using the methods found in [3]. This paper was developed by my tutor and is the initial inspiration behind the thesis, for proof of the theorems that will follow I strongly recommend looking at their article, or it can also be found in the thesis.

Theorem 6.1. *Let $f \in A^0(X)$ be a function so that $df \in A^1(X)$ and $\Delta f \in A^0(X)$ are well defined. For every $\epsilon > 0$ exists a finite subset $L_f(\epsilon) \subset \Lambda$ that only depends of $||f||$, $||df||$, $||\Delta f||$ and ϵ so that*

$$||f - f^{L_f(\epsilon)}|| \leq \epsilon ||f||. \tag{5}$$

Actually, $L_f(\epsilon)$ can be chosen by the preimage $\lambda\colon \Lambda \to \mathbb{R}$ of the compact interval $[L_f^-(\epsilon), L_f^+(\epsilon)]$, where

$$L_f^\pm(\epsilon) = \frac{||df||^2 \pm \epsilon^{-1}\sqrt{||\Delta f||^2||f||^2 - ||df||^4}}{||f||^2}. \tag{6}$$

Theorem 6.2. *Assume we already computed the Fourier coefficients of f for a given subset $I \subset \Lambda$. Then the inequality (5) is also satisfied for the new subset of coefficients*

$$L_f(\epsilon, I) = I \cup L_{f^{\Lambda \setminus I}} \left(\frac{\epsilon ||f||}{||f^{\Lambda \setminus I}||} \right). \tag{7}$$

These formulas will be used in the programs that calculate and display the Fourier coefficients, in order to see their performance on a real scenario, and the results will be discussed at the end.

7. Formulas for computational use

In this section we will explain the formulas implemented in the program to calculate the Fourier series for the case $L^2(\mathbb{S}^2)$. For our particular case we will suppose that we have a triangulation $\{T_i\}_{i=0}^N$ of a star-shaped surface S of strictly positive radius. We will express this surface by its radius $r \colon \mathbb{S}^2 \to \mathbb{R}^+$ given any point of the sphere. To compute the Fourier coefficients for this function we will need to apply (2), and the expression will be as follows,

$$r_\ell^m = \int_{\mathbb{S}^2} r\, Y_\ell^m \, dV_g = \sum_{i=0}^{N} \int_{T_i^p} r\, Y_\ell^m \, dV_g,$$

where $T_i^p \subset \mathbb{S}^2$ is the triangle T_i projected onto the unit sphere. If the triangulation is fine enough, we can approximate that Y_ℓ^m is constant over the entire surface of the triangle, and since the mean radius of the triangle is the radius in the barycenter, we will rewrite the prior expression as

$$r_\ell^m \approx \sum_{i=0}^{N} ||\overline{T}_i||\, Y_\ell^m(\overline{T}_i^p) A(T_i^p),$$

where $\overline{T}_i$ is the barycenter of the triangle T_i, and $A(T_i^p)$ is the area of the spherical triangle T_i^p.

Now we will look at how to evaluate every term of this formula. Starting with $||\overline{T}||$, this is just the norm of the mean of the three $v_0, v_1, v_2 \in \mathbb{R}^3$ points that make the triangle, $||\overline{T}|| = ||\frac{v_0+v_1+v_2}{3}||$.

Then to calculate the area of the spherical triangle $A(T^p)$ we will use the formula $A = \alpha_0 + \alpha_1 + \alpha_2 - \pi$, where α_i are the angles of the spherical triangle. To find these angles we can use the tangent lines to the sphere $u = \frac{(v_i \times v_j) \times v_i}{||(v_i \times v_j) \times v_i||}$, and find the angles through the scalar product between them.

Finally we have to compute the spherical harmonic at the barycenter of the projected triangle. In this case we will explain how the program computes (3) for any given point $(x, y, z) \in \mathbb{S}^2 \subset \mathbb{R}^3$. To compute the trigonometric functions $\cos(m\varphi)$ and $\sin(m\varphi)$ we will use the fact that we know $\cos\varphi = \frac{x}{\sqrt{1-z^2}}$ and $\sin\varphi = \frac{y}{\sqrt{1-z^2}}$ and the Chebyshev polynomial, that give us

$$T_n(\cos\theta) = \cos(n\theta),$$
$$U_{n-1}(\cos\theta)\sin\theta = \sin(n\theta).$$

To compute them we will use the following recurrences,

$$\begin{aligned}
T_0(x) &= 1,\\
T_1(x) &= x,\\
T_{n+1}(x) &= 2xT_n(x) - T_{n-1}(x),\\
U_0(x) &= 1,\\
U_1(x) &= 2x,\\
U_{n+1}(x) &= 2xU_n(x) - U_{n-1}(x).
\end{aligned}$$

For further information about the Chebyshev polynomials and its recurrences you can check [1, Chapter 22], or [4, Chapter 18].

To compute the associated Legendre polynomials, we will use the following recurrences, which can be found in the cited references,

$$P_{\ell+1}^{\ell+1}(x) = -(2\ell-1)\sqrt{1-x^2}P_\ell^\ell(x), \tag{8}$$

$$P_{\ell+1}^{\ell}(x) = x(2\ell+1)P_\ell^\ell(x), \tag{9}$$

$$(\ell-m+1)P_{\ell+1}^m(x) = (2\ell+1)xP_\ell^m(x) - (\ell+m)P_{\ell-1}^m(x). \tag{10}$$

More precisely we will use (8) to compute until P_m^m, then (9) to compute P_{m+1}^m, and finally (10) to compute until P_ℓ^m, using the $\cos\theta = z$ that we already know.

Now we have a precise way of calculating every term of the Fourier series, since we just need to repeat these calculations for every triangle of our triangulation and we will get the Fourier coefficient, and this way is how it has been implemented in our program. Finally to plot the surface we just apply (1),

$$\begin{aligned}
S &= \{r^L(\varphi,\theta)\Phi(\varphi,\theta) \mid (\varphi,\theta) \in (0,2\pi)\times(0,\pi)\},\\
r^L(\varphi,\theta) &= \sum_{(\ell,m)\in L} r_\ell^m Y_\ell^m(\varphi,\theta).
\end{aligned}$$

It is also interesting to see how we compute the formula for the estimated L^2-error given the triangulation. As seen in the formulas (6) and (7) all we need to compute are the norms squared. For the first norm we will use the following formula,

$$||r||^2 = \int_{\mathbb{S}^2} r^2\, dV_g = \sum_{i=0}^N \int_{T_i^p} r^2\, dV_g \approx \sum_{i=0}^N ||\overline{T}_i||^2 A(T_i^p).$$

For the norm of the differential we will use the scalar product as described giving us the following expression,

$$||dr||^2 = \int_{\mathbb{S}^2} g^*(dr,dr)\, dV_g = \int_{\mathbb{S}^2} \left[r_\theta^2 + \left(\frac{r_\varphi}{\sin\theta}\right)^2\right] dV_g,$$

which can be approximated as shown in the thesis by the following sum,

$$||dr||^2 \approx \sum_{i=0}^N ||\overline{T}_i||^2 \left[\frac{||\overline{T}_i||^2}{(N\cdot\overline{T}_i)^2} - 1\right] A(T_i^p).$$

We are just missing an expression for the Laplacian norm. In this case we will be using the formula discussed in [6], this will give us a value for the Laplacian in every vertex of the triangulation.

Let $\{v_i\}_{i=1}^V$ be the set of vertex of our triangulation, we denote $\{p_i\}_{i=1}^V$ their projections on the unit sphere so that $p_i = v_i^p$. For every vertex p_i with n neighbors, we denote $N(i) = i_1, \dots, i_n$ the set of index of the neighbor vertex of p_i. We will assume they are ordered counterclockwise as seen from outside the sphere above p_i. Then we can approximate the Laplacian in this spot as

$$\Delta_{\mathcal{M}} r(p_i) = \frac{4\sum_{j\in N(i)}(\cot\alpha_{ij} + \cot\beta_{ij})(||v_j|| - ||v_i||)}{\sum_{j\in N(i)}(\cot\alpha_{ij} + \cot\beta_{ij})||p_j - p_i||^2},$$

where α_{ij} and β_{ij} are the angles of the adjacent triangles to the segment p_ip_j.

Finally since we want to compute $||\Delta r||$. We will consider that each one of our vertex takes a region $s(p_i) \subset \mathbb{S}^2$ equivalent to one third of the area of the spherical triangles surrounding it. Therefore the area of all the triangles will be equally shared between the vertex, and we obtain the expression

$$\begin{aligned} ||\Delta r||^2 &= \int_{S^2} (\Delta r)^2\, dV_g \approx \sum_{i=0}^V \int_{s(p_i)} (\Delta_{\mathcal{M}} r)^2\, dV_g \\ &= \sum_{i=0}^V \left[\frac{4\sum_{j\in N(i)}(\cot\alpha_{ij} + \cot\beta_{ij})(||v_j|| - ||v_i||)}{\sum_{j\in N(i)}(\cot\alpha_{ij} + \cot\beta_{ij})||p_j - p_i||^2}\right]^2 \frac{1}{3}\sum_{j\in N(i)} A(T_{ij}^p). \end{aligned}$$

8. Conclusion and results

As mentioned previously this thesis was built around the idea of creating a program that can replicate all the formulas explained here and can show some interesting results. For the interested reader this program does exist and can be found at `https://github.com/MiquelNasarre/FourierS2`.

It is satisfying to see that all these formulas actually work and can produce some interesting results. This can be seen in Figure 3, that shows the spherical harmonics as depicted by the program, and Figure 1, that shows a basic example of the program's functionality.

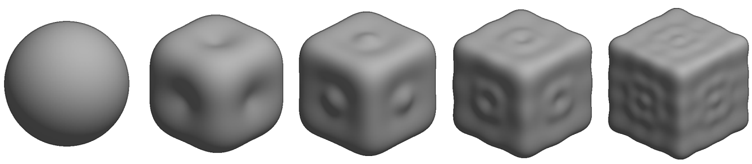

Figure 1: Program trying to recreate a cube triangulation with ℓ from 0 to 4.

Also this program allows us to see some clear limitations of the formulas. For example if you try to go too deep and your triangulation is not fine enough the approximations to calculate the coefficients will not be as good, giving you some weird looking shapes in the process, as seen in the middle shape of Figure 2. This limitation though can be easily solved by dividing the triangles in the triangulation, as shown by the last shape of Figure 2, where the shape is visibly better defined and the L^2-error is lower.

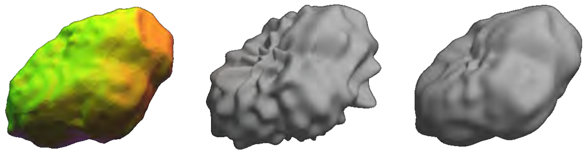

Figure 2: Program creating the Fourier series $\ell \leq 20$ of `example.dat` without the subdivision and with four subdivisions of the triangles.

Another limitation that can not be solved easily is the error formulas, due to the amount of approximations involving the entire process these formulas have proven not to be very reliable for the case in $L^2(\mathbb{S}^{\neq})$ although they have shown great results for the common Fourier series.

Overall I am very satisfied of the results obtained by the program as well as all the mathematical background developed in the thesis to back it up, I hope this article is useful for someone who decides to undertake a similar case of study in the future.

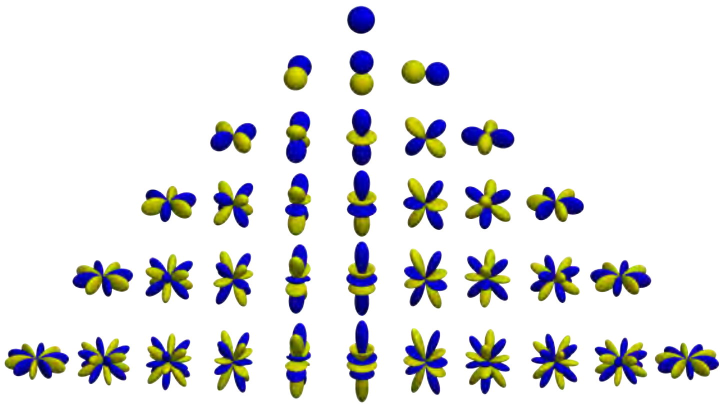

Figure 3: Spherical harmonics as shown by the program with ℓ from 0 to 5.

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