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Ideals of p^e-th roots of plane curves in positive characteristic

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Resum *(CAT)*
Una estratègia per estudiar varietats algebraiques és construir invariants algebraics que mesurin les seves singularitats. Sobre els nombres complexos, destaquen els ideals multiplicadors i els nombres de salt. En característica positiva, les seves contraparts són els ideals de test i els F-nombres de salt. En aquest projecte, calculem els ideals de test i F-nombres de salt de corbes planes quasi-homogènies, així com de les seves deformacions a nombre de Milnor constant, per una quantitat infinita de característiques $p > 0$. En aquests casos, veiem que els ideals de test són la reducció mòdul p dels ideals multiplicadors.

Abstract *(ENG)*
A common approach to studying algebraic varieties is through algebraic invariants that measure their singularities. Over the complex numbers, a celebrated example of such invariants include the multiplier ideals and the jumping numbers. In positive characteristic, their counterparts are the test ideals and F-jumping numbers. In this work, we compute the test ideals and F-jumping numbers of quasi-homogeneous plane curves, as well as their one-monomial constant Milnor number deformations, for infinitely many characteristics $p > 0$. In these cases, we see that the test ideals are the modulo p reduction of the multiplier ideals.

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1. Introduction

A challenge that lies at the heart of modern algebraic geometry is the classification of algebraic varieties which, in particular, encompasses the characterization of their singularities. The most common approach to this problem is to construct algebraic and geometric invariants to quantify the singularities.

In birational algebraic geometry over the complex numbers, or more generally, over fields of characteristic zero, one can take advantage of the existence of a resolution of singularities to construct invariants. A celebrated example of such invariants is the family of multiplier ideals. Given a hypersurface defined by the vanishing locus of a polynomial f, the multiplier ideals $\mathcal{J}(f^{\lambda})$ form a family of ideals indexed by nonnegative real numbers $\lambda \in \mathbb{R}_{\geq 0}$. These give a descending chain of ideals $\mathcal{J}(f^{\lambda}) \supseteq \mathcal{J}(f^{\mu})$ whenever $\lambda \leq \mu$, which in addition is right-semicontinuous, meaning that $\mathcal{J}(f^{\lambda}) = \mathcal{J}(f^{\lambda+\varepsilon})$ for some $\varepsilon > 0$. The values $\lambda > 0$ where the chain jumps, that is, $\mathcal{J}(f^{\lambda-\varepsilon}) \supsetneq \mathcal{J}(f^{\lambda})$ for arbitrarily small $\varepsilon > 0$, are known as the jumping numbers of f. The smallest jumping number among them is called the log-canonical threshold. By means of the resolution of singularities, one can show this set rational and discrete. References are made to [6].

The multiplier ideals and jumping numbers encode the singularities of the hypersurface determined by f in subtle ways. Suppose, for instance, that the vanishing locus of f is a curve C in the complex plane with a singularity at the origin. If one deforms C into a new curve C' while preserving the analytic type of the singularity, then the entire family of multiplier ideals remains unchanged. On the contrary, if the deformation only preserves the topological type, then the jumping numbers still agree, although the multiplier ideals may, in general, differ.

Over fields of positive characteristic $p > 0$, there is no resolution of singularities available for varieties of arbitrary finite dimension. In this setting, the Frobenius endomorphism, or p-th power map, serves as a substitute tool. The test ideals, which play the role of the multiplier ideals, were introduced by Hochster and Huneke as an auxiliary tool in tight closure theory [5], and were later refined by Hara and Yoshida [4]. We shall adopt the construction of Blickle, Mustaţă, and Smith (see Definition 2.9), which generalizes earlier definitions [3]. In brief, the test ideals $\tau(f^{\lambda})$ of a polynomial f are a nested, right-semicontinuous family of ideals indexed over the nonnegative real numbers $\lambda \in \mathbb{R}_{\geq 0}$. The spots where the chain of test ideals "jumps" are the F-jumping numbers of f, the smallest of which is the F-pure threshold.

It is a well-established fact due to Mustaţă, Takagi, and Watanabe, that if f is a polynomial defined over the integers, the log-canonical threshold of f can be recorvered from the F-pure thresholds of the reductions f_p of f modulo a prime p, as $p \to \infty$ [9]. A profound conjecture in arithmetic geometry, the weak ordinarity conjecture, proposes a further connection, namely, that the test ideals of the reductions f_p can be calculated by reducing the multiplier ideals modulo p, for all primes in a Zariski-dense set [8]. In this sense, the theories of multiplier ideals in characteristic zero and test ideals in positive characteristic are closely analogous.

Test ideals and F-jumping numbers are notoriously difficult to compute. In the few cases where explicit descriptions are known—such as elliptic curves, diagonal hypersurfaces, determinantal ideals of maximal minors, or ideals invariant under the action of a subgroup of a linear group, the calculations rely on the arithmetic or combinatorial properties of the variety. A naive yet effective approach to obtain $\tau(f^{\lambda})$ is to calculate the ideals p^e-th roots of f, denoted $\mathcal{C}_R^e \cdot f^n$ (see Definition 2.5), for a fixed integer $e \geq 0$. As $n \geq 0$ ranges over the natural numbers integers, one obtains descending chain of ideals

$$\mathcal{C}_R^e \cdot f^0 \supseteq \mathcal{C}_R^e \cdot f \supseteq \mathcal{C}_R^e \cdot f^2 \supseteq \cdots \supseteq \mathcal{C}_R^e \cdot f^n \supseteq \mathcal{C}_R^e \cdot f^{n+1} \supseteq \cdots$$

which, in essence, contains all the test ideals of f, and codifies the F-jumping numbers.

In this work, we begin by studying quasi-homogeneous plane curves C over a perfect field K of characteristic $p > 0$, that is, curves in K^2 given as the vanishing locus of a polynomial of the form $f = x^a + y^b$, with $a, b \geq 2$. For these, we observe the ideals of p^e-th roots, and consequently the test ideals are monomial ideals for sufficiently big characteristics $p \gg 0$. To determine the F-jumping numbers, we pose a linear integer programming problem, and provide its solution.

We then turn to deformations C' of the original curve C, which are curves given as the zeros of polynomials $g = f + \sum_i t_i x^{\alpha_i} y^{\beta_i}$. We restrict ourselves, however, to one-monomial deformations $g = f + tx^\alpha y^\beta$. From the algebro-geometric standpoint, it is most natural to consider deformations that preserve the singularity type of C at the origin, namely, constant Milnor number deformations. In this setting, we again find that the p^e-th roots and test ideals are monomial.

For all sufficiently large primes such that $p \equiv 1 \pmod{ab}$, we describe explicitly the chains of p^e-th roots, test ideals, and F-jumping numbers, and observe that they coincide for f and g. Finally, we note that the test ideals in this setting arise as reductions modulo p of the corresponding multiplier ideals.

2. Invariants of singularities in characteristic $p > 0$

Throughout, let R denote a ring of characteristic $p > 0$, and let $F\colon R \to R$, $f \mapsto f^p$, be the Frobenius endomorphism of R. For a nonnegative integer $e \geq 0$, the e-th iterated Frobenius is the endomorphism $F^e\colon R \to R$, $f \mapsto f^{p^e}$. In this section we introduce invariants of singularities in positive characteristic of interest to us. Often, these are referred to as F-invariants for they originate from the action of the Frobenius on R.

Restriction of scalars along F^e endows R with an exotic R-module structure denoted $F^e_* R$. Its elements are written as $F^e_* x$ for $x \in R$. As an abelian group with respect to addition, $F^e_* R$ is isomorphic to R. The action of R on $F^e_* R$ is given by restriction of scalars: $r \cdot F^e_* x := F^e_*(r^{p^e} x)$, for $r \in R$, $F^e_* x \in F^e_* R$. A Noetherian ring R of characteristic $p > 0$ is said to be F-finite provided $F^e_* R$ is a finite R-module for some $e \geq 1$ (equiv. all $e \geq 1$).

Example 2.1. Let $R = K[x_1, \dots, x_n]$ be a polynomial ring over a field K of characteristic $p > 0$. If K is perfect, i.e. the Frobenius $F\colon K \xrightarrow{\simeq} K$ is an automorphism of K, then $F^e_* R$ splits as

$$F^e_* R \simeq \bigoplus_{0 \leq i_1, \dots, i_n < p^e} R\, F^e_* x_1^{i_1} \cdots x_n^{i_n},$$

therefore R is an F-finite ring, and $F^e_* R$ is a finite free module with basis $\{F^e_* x_1^{i_1} \cdots x_n^{i_n} \mid 0 \leq i_1, \dots, i_n < p^e\}$. In the sequel, we will refer to this as the standard basis of $F^e_* R$.

2.1. Frobenius powers and p^e-th roots of ideals

Let R be a regular F-finite ring of characteristic $p > 0$.

Definition 2.2. Let I be an ideal of R. For an integer $e \geq 0$, the e-th Frobenius power of I is the ideal

$$I^{[p^e]} = F^e(I)R = (f^{p^e} \mid f \in I).$$

Remark 2.3. One checks that if $I = (f_\lambda \mid \lambda \in \Lambda)$ is a generating set for I, then $I^{[p^e]} = (f_\lambda^{p^e} \mid \lambda \in \Lambda)$, hence $I^{[p^e]} \subseteq I^{p^e}$. The reverse containment holds when $I = (f)$ is principal, that is, $(f)^{[p^e]} = (f)^{p^e}$, thus Frobenius powers and regular powers coincide for principal ideals.

A sort of "converse" operation to the Frobenius power is the ideal of p^e-th roots of an ideal I. These were introduced in [1] in the principal case under the notation $I_e(f)$, and later on exploited in [3] to give an alternative definition of the test ideals (see Section 2.2), using the notation $I^{[1/p^e]}$.

Definition 2.4. A Cartier operator of level $e \geq 0$ is an R-linear map $F_*^e R \to R$. The set of Cartier operators of level $e \geq 0$ has a natural R-module structure, which we denote by $\mathcal{C}_R^e := \mathrm{Hom}_R(F_*^e R, R)$.

Definition 2.5. Let I be an ideal of R. For an integer $e \geq 0$, the ideal of p^e-th roots of I is the ideal

$$\mathcal{C}_R^e \cdot I = (\varphi(F_*^e f) \mid \varphi \in \mathcal{C}_R^e,\ f \in I).$$

Remark 2.6. If R is a regular and F-finite, $\mathcal{C}_R^e \cdot I$ is characterized as the smallest ideal of R in the sense of inclusion such that $I \subseteq (\mathcal{C}_R^e \cdot I)^{[p^e]}$.

Proposition 2.7 ([1], [3, Proposition 2.5])**.** *Suppose that $F_*^e R$ is a free R-module with basis $F_*^e x_1, \dots, F_*^e x_n$. For an ideal $I = (f_1, \dots, f_m)$ of R, let*

$$F_*^e f_i = \sum_{1 \leq j \leq n} f_{ij} F_*^e x_j$$

be the expression in the basis of f_i, $i = 1, \dots, m$. Then $\mathcal{C}_R^e \cdot I = (f_{ij} \mid 1 \leq i \leq m,\ 1 \leq j \leq n)$.

We collect below a few facts about p^e-th roots that will be useful later on; for a proof, we refer the reader to [3, Lemma 2.4].

Lemma 2.8. *Let I, J be ideals of R, and $d, e \geq 0$ be nonnegative integers.*

(i) If $I \subseteq J$, then $\mathcal{C}_R^e \cdot I \subseteq \mathcal{C}_R^e \cdot J$.

(ii) One has that $J \cdot (\mathcal{C}_R^e \cdot I) = \mathcal{C}_R^e \cdot (I \cdot J^{[p^e]})$.

(iii) One has that $\mathcal{C}_R^e \cdot I = \mathcal{C}_R^{d+e} \cdot I^{[p^d]}$. In particular, if $I = (f)$ is principal, then $\mathcal{C}_R^e \cdot f = \mathcal{C}_R^{d+e} \cdot f^{p^d}$.

2.2. Test ideals, F-jumping numbers, and ν-invariants

Let R denote regular F-finite ring of characteristic $p > 0$. We now introduce the generalized test ideals of an ideal in R, as defined in [3], along with their associated invariants. Throughout, we denote by $\lceil x \rceil$ the ceiling of a real number.

Definition 2.9. Fix an ideal I of R. The test ideal of I with exponent $\lambda \in \mathbb{R}_{\geq 0}$ is

$$\tau(I^\lambda) = \bigcup_{e \geq 0} \mathcal{C}_R^e \cdot I^{\lceil \lambda p^e \rceil}.$$

Remark 2.10. It can be shown that the p^e-th roots appearing on the right-hand side give an ascending chain of ideals, which eventually stabilizes because R is Noetherian, therefore $\tau(I^\lambda) = \mathcal{C}_R^e \cdot I^{\lceil \lambda p^e \rceil}$ for $e \gg 0$.

Since p^e-th roots preserve inclusions (Lemma 2.8), so do test ideals, that is, $\tau(I^\lambda) \supseteq \tau(I^\mu)$ whenever $\lambda \leq \mu$. It follows the test ideals give a descending family of ideals obtained as λ ranges over the nonnegative real numbers. This chain is right semi-continuous in the following sense:

Theorem 2.11 ([9, Remark 2.12], [3, Corollary 2.16, Theorem 3.1])**.** *Let I be an ideal of R.*

(i) For each $\lambda \geq 0$, there exists $\varepsilon > 0$ such that $\tau(I^\lambda) = \tau(I^{\lambda+\varepsilon})$.

(ii) There exist real numbers $\lambda > 0$ such that $\tau(I^{\lambda-\varepsilon}) \supsetneq \tau(I^\lambda)$ for all $\varepsilon > 0$.

Definition 2.12. A positive real number $\lambda > 0$ is an F-jumping number of an ideal I of R provided

$$\tau(I^{\lambda-\varepsilon}) \supsetneq \tau(I^\lambda), \quad \text{for all } \varepsilon > 0.$$

The F-pure threshold of I, written $\mathrm{fpt}(I)$, is defined as the infimum among the F-jumping numbers of I. It is characterized by $\mathrm{fpt}(I) = \sup\{\lambda > 0 \mid \tau(I^\lambda) = R\}$.

Theorem 2.13 (Skoda's theorem, [3, Proposition 2.25])**.** *Suppose that I is an ideal generated by n elements. Then $\tau(I^\lambda) = I \cdot \tau(I^{\lambda-1})$ for every real number $\lambda \geq n$.*

The result below shows that the log-canonical threshold of a polynomial with integer coefficients can be recovered from the F-pure thresholds of the reductions modulo p:

Theorem 2.14 ([9, Theorem 3.4])**.** *Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial with rational coefficients. For a prime number $p > 0$, let $f_p \in \mathbb{F}_p[x_1, \ldots, x_n]$ be the reduction of f modulo p. Then*

$$\mathrm{lct}(f) = \lim_{p\to\infty} \mathrm{fpt}(f_p).$$

It is conjectured that the statement above generalizes, in a sense, to multiplier and test ideals. Before announcing it, let us remark that if s is a Zariski-closed point in the spectrum of a finitely generated $\mathbb{Z}$-algebra A, that is, s is a maximal ideal of A, then the quotient A/sA has positive characteristic.

Conjecture 2.15 (Weak ordinarity conjecture, [8, Conjecture 1.2])**.** *Let I be an ideal in the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$. Suppose that I can be generated by elements in a finitely generated $\mathbb{Z}$-subalgebra A of $\mathbb{C}[x_1, \ldots, x_n]$. Then there exists a Zariski-dense subset S of $\operatorname{Spec} A$ consisting of closed points such that*

$$\mathcal{J}(I^\lambda)_s = \tau(I_s^\lambda), \quad \textit{for all } \lambda \geq 0,$$

for every $s \in S$, where I_s^λ, $\mathcal{J}(I^\lambda)_s$ denote the image under $A \to A/sA$ of I^λ, and $\mathcal{J}(I^\lambda)$, respectively.

Closely related to the F-jumping numbers are the F-thresholds introduced in [9]. Their definition is based on a different family of invariants, namely, the ν-invariants, which are of interest by themselves.

Definition 2.16. Let I, J be ideals of R such that $I \subseteq \sqrt{J}$, where $\sqrt{J}$ denotes the radical of J. The ν-invariant of level $e \geq 0$ of I with respect to J is

$$\nu_I^J(p^e) = \max\{n \geq 0 \mid I^n \subseteq J^{[p^e]}\}.$$

Let us denote by $\nu_I^\bullet(p^e)$ the set of ν-invariants of level $e \geq 0$ of I.

Definition 2.17. Let I, J be ideals of R such that $I \subseteq \sqrt{J}$. The F-threshold of I with respect to J is

$$c^J(I) = \lim_{e\to\infty} \frac{\nu_I^J(p^e)}{p^e}.$$

Theorem 2.18 ([3, Corollary 2.30, Theorem 3.1])**.** *The set of F-jumping numbers of an ideal I coincides with the set of F-thresholds $c^J(I)$ of I obtained as J ranges over all the ideals of R satisfying $I \subseteq \sqrt{J}$. In particular, both sets are rational and discrete.*

In computing F-thresholds, different ideals J, J' containing I in their radical may give raise to the same F-threshold. Instead, however, one can look at the spots where the chain of ideals below jumps:

$$\cdots \supseteq \mathcal{C}_R^e \cdot I^{r-1} \supseteq \mathcal{C}_R^e \cdot I^r \supseteq \mathcal{C}_R^e \cdot I^{r+1} \supseteq \cdots .$$

Definition 2.19 ([10, Proposition 4.2])**.** The set of ν-invariants of level $e \geq 0$ of an ideal I of R is

$$\nu_I^\bullet(p^e) = \{r \geq 0 \mid \mathcal{C}_R^e \cdot I^r \supsetneq \mathcal{C}_R^e \cdot I^{r+1}\}.$$

2.3. Chains of p^e-th roots and r-invariants

Through this section, let $R = K[x_1, \dots, x_d]$ denote a polynomial ring over a field K of characteristic $p > 0$. We are interested in obtaining the test ideals $\tau(f^\lambda)$ of a polynomial $f \in R$. Skoda's theorem (see Theorem 2.13) shows that $\tau(f^\lambda) = (f)\tau(f^{\lambda-1})$ for every real number $\lambda \geq 1$, hence it suffices to look at test ideals with $0 < \lambda < 1$. In this case, by Remark 2.10, one has that $\tau(f^\lambda) = \mathcal{C}_R^e \cdot f^r$ for some integer $r \leq p^e$. Aside from test ideals, we are keen on p^e-th roots $\mathcal{C}_R^e \cdot f^n$. By writing n uniquely as $n = sp^e + r$, with $s \geq 0$, $0 \leq r < p^e$, it follows from Lemma 2.8 that $\mathcal{C}_R^e \cdot f^n = (f)^s \cdot \mathcal{C}_R^e \cdot f^r$. Altogether, this shows it is enough to consider the ideals in chains of the form

$$R = \mathcal{C}_R^e \cdot f^0 \supseteq \mathcal{C}_R^e \cdot f \supseteq \mathcal{C}_R^e \cdot f^2 \supseteq \cdots \supseteq \mathcal{C}_R^e \cdot f^{p^e-2} \supseteq \mathcal{C}_R^e \cdot f^{p^e-1}. \tag{1}$$

Definition 2.20. We refer to (1) as the chain of ideals of p^e-th roots of f.

Lemma 2.21. *For a polynomial $f \in R$, one has that $\nu_f^\bullet(p^e) = (\nu_f^\bullet(p^e) \cap [0, p^e)) + p^e\mathbb{Z}_{\geq 0}$.*

Notation 2.22. *If $\underline{u} = (u_1, \dots, u_d) \in \mathbb{Z}_{\geq 0}^d$ is a multi-index, we let $\underline{x}^{\underline{u}}$ be the monomial $\underline{x}^{\underline{u}} = x_1^{u_1} \cdots x_d^{u_d}$.*

Perhaps, the most straightforward way to detect a jump $\mathcal{C}_R^e \cdot f^r \supsetneq \mathcal{C}_R^e \cdot f^{r+1}$ in chain (1) is to test if a monomial $\underline{x}^{\underline{u}}$ in $\mathcal{C}_R^e \cdot f^r$ drops from $\mathcal{C}_R^e \cdot f^{r+1}$, which means r is a ν-invariant attached to f and the monomial $\underline{x}^{\underline{u}}$. While this technique is standard in the field, to the best of our knowledge, the invariant "r" has not been assigned a name in the literature. We therefore introduce the following definition:

Definition 2.23. We define the r-invariant of level $e \geq 0$ of f with respect to a monomial $\underline{x}^{\underline{u}}$ by

$$r_R^e(f; \underline{x}^{\underline{u}}) = \sup\{n \in \mathbb{Z} \mid \underline{x}^{\underline{u}} \in \mathcal{C}_R^e \cdot f^n\}.$$

Lemma 2.24. *Let $f \in R$ be a polynomial. Suppose that f is not a monomial.*

(i) For all monomials $\underline{x}^{\underline{u}}$ in R, $0 \leq r_R^e(f, \underline{x}^{\underline{u}}) \leq p^e - 1$.

(ii) Every r-invariant of f is a ν-invariant.

When every ideal in the chain of p^e-th roots of f is monomial, the converse to Lemma 2.24 holds. In spite of how restrictive this latter condition may seem, we will come across it in Section 3.

Lemma 2.25. *Let $f \in R$ be a polynomial that is not a monomial. Suppose that every ideal in the chain of p^e-th roots of f is monomial.*

(i) One has that $\nu_f^\bullet(p^e) \cap [0, p^e) = \{r_R^e(f; \underline{x}^{\underline{u}}) \mid \underline{x}^{\underline{u}}\}$.

(ii) For an integer $0 \le n < p^e$, one has that $\mathcal{C}_R^e \cdot f^n = (\underline{x}^{\underline{u}} \mid r_R^e(f; \underline{x}^{\underline{u}}) \le n)$.

3. Ideals of p^e-th roots of plane curves

In this section, we describe the invariants previously introduced, for quasi-homogeneous plane curves defined over perfect fields of characteristic $p > 0$, for infinitely many primes, and their one-monomial constant Milnor number deformations. Throughout, let $\lfloor x \rfloor$ and $\lceil x \rceil$ be the floor and ceil functions, respectively.

Remark 3.1. Let $R = K[x_1, \dots, x_d]$ be a polynomial ring over a perfect field K of characteristic $p > 0$, so $F_*^e R$ is a free R-module with standard basis $\{F_*^e x_1^{i_1} \cdots x_d^{i_d} \mid 0 \le i_1, \dots, i_d < p^e\}$ (Example 2.1). Given a monomial $x_1^{u_1} \cdots x_d^{u_d}$, write each exponent uniquely as $u_i = s_i p^e + r_i$, with $s_i \ge 0$, $0 \le r_i < p^e$, for $i = 1, \dots, d$. Then

$$F_*^e(x_1^{u_1} \cdots x_n^{u_d}) = x_1^{s_1} \cdots x_n^{s_d}\, F_*^e(x_1^{r_1} \cdots x_n^{r_d})$$

is the basis expression of $x_1^{u_1} \cdots x_d^{u_d}$. Note that $s_i = \lfloor u_i/p^e \rfloor$, and r_i is the only integer $0 \le r_i < p^e$ with $u_i \equiv r_i \pmod{p^e}$. This calculation extends linearly to polynomials of R.

Definition 3.2. In the setting above, we say the monomials $x_1^{u_1} \cdots x_d^{u_d}$, $x_1^{v_1} \cdots x_d^{v_d}$ appear with the same basis element if $F_*^e x_1^{u_1} \cdots x_d^{u_d}$, $F_*^e x_1^{v_1} \cdots x_d^{v_d}$ lie in the same rank-one free R-submodule of $F_*^e R$ spanned by an element of the standard basis of $F_*^e R$. This is equivalent to $u_i \equiv v_i \pmod{p^e}$, for $i = 1, \dots, d$.

3.1. Ideals of p^e-th roots of quasi-homogeneous plane curves

Definition 3.3. Let K be a field. A quasi-homogeneous plane curve defined over K is the vanishing locus in K^2 of a polynomial of the form $f = x^a + y^b$, where $a, b \ge 2$. For simplicity, we will refer to the binomial $f = x^a + y^b$ as the quasi-homogeneous plane curve.

From now on, we work over the polynomial ring $R = K[x, y]$, with K perfect of characteristic $p > 0$. We remark, however, that all results remain valid upon relaxing the assumption on K to merely being F-finite, i.e. that K/K^{p^e} be a finite extension for some $e > 0$ (equiv. all $e > 0$).

Proposition 3.4. *Let $f = x^a + y^b$ be a quasi-homogeneous plane curve. Suppose that p does not divide a or b. For every integer $0 \le n < p^e$, one has that*

$$\mathcal{C}_R^e \cdot f^n = \left(x^{\lfloor ai/p^e \rfloor} y^{\lfloor bj/p^e \rfloor} \;\middle|\; i + j = n \text{ and } \binom{n}{i, j} \not\equiv 0 \pmod{p} \right).$$

In particular, every ideal in the chain of p^e-th roots of f is monomial.

Remark 3.5. In view of Proposition 3.4, $\mathcal{C}_R^e \cdot f^n$ contains the monomial $x^u y^v$, if and only if there exists a pair (i,j) of nonnegative integers such that:

$$\left\lfloor \frac{ai}{p^e} \right\rfloor \leq u, \quad \text{and} \quad \left\lfloor \frac{bj}{p^e} \right\rfloor \leq v, \quad \text{and} \quad \binom{n}{i,j} \not\equiv 0 \pmod{p}, \quad \text{and} \quad i+j=n.$$

The first two conditions are equivalent to $ai \leq (u+1)p^e - 1$, and $bj \leq (v+1)p^e - 1$, respectively. As a result, the r-invariant $r_R^e(f; x^u y^v)$ is the solution to the following linear integer programming problem:

$$\begin{cases} \text{maximize:} & i+j, \\ \text{subject to:} & ai \leq (u+1)p^e - 1, \\ & bj \leq (v+1)p^e - 1, \\ & \binom{i+j}{i,j} \not\equiv 0 \pmod{p}, \\ & i,j \in \mathbb{Z}_{\geq 0}. \end{cases} \tag{P1}$$

A solution $\geq p^e$ has no meaning by Lemma 2.24, hence it should be thought of as $x^u y^v \in \mathcal{C}_R^e \cdot f^{p^e-1}$.

One can give general bounds for the optimum of (P1). To obtain a solution, however, it is helpful to make assumptions on the congruence class of p modulo ab. In what follows, we provide the solution under such additional assumption. Later on, in Section 3.3, we study the consequences on the F-invariants.

Lemma 3.6. *Let $f = x^a + y^b$ be a quasi-homogeneous plane curve. Suppose that $p \equiv 1 \pmod{ab}$. The r-invariant $r_R^e(f; x^u y^v)$ of a monomial $x^u y^v$ is*

$$r_R^e(f; x^u y^v) = \begin{cases} \left(\frac{u+1}{a} + \frac{v+1}{b}\right)(p^e - 1) & \text{if } bu + av < ab - a - b, \\ p^e - 1 & \text{if } bu + av \geq ab - a - b. \end{cases}$$

3.2. One-monomial deformations of quasi-homogeneous plane curves

Let $C \subseteq K^2$ be a plane curve given as the zero locus of a polynomial $h \in K[x,y]$. Suppose C passes through the origin. Then the Milnor number of C is defined as $\mu = \dim_K K[x,y]/(\partial_x h, \partial_y h)$. When $K = \mathbb{C}$, and h is a quasi-homogeneous plane curve $x^a + y^b$, the Milnor number determines the topological type of the singularity of C at the origin under deformations, in the following sense:

Theorem 3.7 ([7]). *Let $f = x^a + y^b$ be a quasi-homogeneous plane curve defined over $\mathbb{C}$, and g a deformation $g = x^a + y^b + t_1 x^{\alpha_1} y^{\beta_1} + \cdots + t_n x^{\alpha_n} y^{\beta_n}$, with $t_1, \ldots, t_n \in \mathbb{C}$. Suppose every deformation monomial $x^{\alpha_i} y^{\beta_i}$ on which g is supported (i.e. $t_i \neq 0$) satisfies $0 \leq \alpha_i < a-1$, $0 \leq \beta_i < b-1$, and $a\beta_i + b\alpha_i > ab$. Then f and g have the same Milnor number.*

Definition 3.8. A constant Milnor number deformation, or μ-constant deformation of a quasi-homogeneous plane curve $f = x^a + y^b$ defined over K, is the vanishing locus in K^2 of a polynomial of the form

$$g = x^a + y^b + \sum_{i=1}^{n} t_i x^{\alpha_i} y^{\beta_i}, \quad \text{where } t_i \in K,$$

with $0 \leq \alpha_i < a-1$, $0 \leq \beta_i < b-1$, and $a\beta_i + b\alpha_i > ab$, for $i = 1, \ldots, n$.

Hereinafter, we consider one-monomial μ-constant deformations of quasi-homogeneous plane curves defined over a perfect field K of characteristic $p > 0$, thus we work over the polynomial ring $R = K[x, y]$.

Proposition 3.9. *Let $g = x^a + y^b + tx^\alpha y^\beta$ be a μ-constant deformation of $f = x^a + y^b$. Suppose that p does not divide a or b, and $p > a\beta + b\alpha - ab$. For every integer $0 \le n < p^e$, one has that*

$$\mathcal{C}_R^e \cdot g^n = \left(x^{\lfloor (ai+\alpha k)/p^e \rfloor} y^{\lfloor (bj+\beta k)/p^e \rfloor} \;\middle|\; i+j+k = n \text{ and } \binom{n}{i,j,k} \not\equiv 0 \pmod p \right).$$

Proposition 3.10. *Let $g = x^a + y^b + tx^\alpha y^\beta$ be a μ-constant deformation of the quasi-homogeneous plane curve $f = x^a + y^b$. Suppose that p does not divide a or b, and $p > a\beta + b\alpha - ab$. One has that $\mathcal{C}_R^e \cdot f^n \subseteq \mathcal{C}_R^e \cdot g^n$ for every integer $0 \le n < p^e$.*

Remark 3.11. By Proposition 3.9, given an integer $0 \le n < p^e$, a monomial $x^u y^v$ is in $\mathcal{C}_R^e \cdot g^n$ if and only if there exists a triple (i, j, k) of nonnegative integers such that:

$$\left\lfloor \frac{ai+\alpha k}{p^e} \right\rfloor \le u, \quad \text{and} \quad \left\lfloor \frac{bj+\beta k}{p^e} \right\rfloor \le v, \quad \text{and} \quad \binom{n}{i,j,k} \not\equiv 0 \pmod p, \quad \text{and} \quad i+j+k = n.$$

One sees the first two conditions are equivalent to $ai + \alpha k \le (u+1)p^e - 1$, and $bj + \beta k \le (v+1)p^e - 1$. It follows that $r_R^e(g; x^u y^v)$ is the solution to the following linear integer programming problem:

$$\begin{cases} \text{maximize:} & i+j+k, \\ \text{subject to:} & ai+\alpha k \le (u+1)p^e - 1, \\ & bj+\beta k \le (v+1)p^e - 1, \\ & \binom{i+j+k}{i,j,k} \not\equiv 0 \pmod p, \\ & i,j,k \in \mathbb{Z}_{\ge 0}. \end{cases} \tag{P2}$$

By Proposition 3.10, a solution of (P2) is bounded below by a solution of (P1). As in Remark 3.5, a solution $\ge p^e$ must be thought of as $x^u y^v \in \mathcal{C}_R^e \cdot g^{p^e-1}$.

Lemma 3.12. *Let $g = x^a + y^b + tx^\alpha y^\beta$ be a μ-constant deformation of a quasi-homogeneous plane curve $f = x^a + y^b$. Suppose that $p \equiv 1 \pmod{ab}$. For a monomial $x^u y^v$, one has that*

$$r_R^e(f; x^u y^v) = r_R^e(g; x^u y^v).$$

3.3. F-invariants of quasi-homogeneous plane curves and deformations

To conclude, we use the r-invariants of quasi-homogeneous plane curves and their one-monomial μ-constant deformations to compute their F-invariants when $p \equiv 1 \pmod{ab}$.

Proposition 3.13. *Let h be either the quasi-homogeneous plane curve $f = x^a + y^b$, or the μ-constant deformation $g = x^a + y^b + tx^\alpha y^\beta$. Suppose that $p \equiv 1 \pmod{ab}$.*

(i) The p^e-th root of h^n, with $0 \leq n < p^e$, is

$$\mathcal{C}_R^e \cdot h^n = \left(x^u y^v \;\middle|\; \frac{u+1}{a} + \frac{v+1}{b} \leq \frac{n}{p^e - 1} \right).$$

(ii) The ν-invariants of h of level e are

$$\nu_h^\bullet(p^e) = \left\{ kp^e + \left(\frac{u+1}{a} + \frac{v+1}{b} \right)(p^e - 1),\, (k+1)p^e - 1 \;\middle|\; bu + av < ab - a - b,\, k \geq 0 \right\}.$$

Theorem 3.14. *Let h be either the quasi-homogeneous plane curve $f = x^a + y^b$, or the μ-constant deformation $g = x^a + y^b + tx^\alpha y^\beta$. Suppose that $p \equiv 1 \pmod{ab}$, and $p > a\beta + b\alpha - ab$.*

(i) The F-jumping numbers of h are

$$\mathrm{FJN}(h) = \left\{ \frac{u+1}{a} + \frac{v+1}{b}, 1 \;\middle|\; bu + av < ab - a - b \right\} + \mathbb{Z}_{\geq 0}.$$

(ii) The test ideal of h with exponent $\lambda \in (0,1)$ is

$$\tau(h^\lambda) = \left(x^u y^v \;\middle|\; \frac{u+1}{a} + \frac{v+1}{b} > \lambda \right).$$

Remark 3.15. For a prime number p sufficiently large with $p \equiv 1 \pmod{ab}$, Proposition 3.13 and Theorem 3.14 show that a quasi-homogenous plane curve and a one-monomial μ-constant deformation have the same chains of p^e-th roots, ν-invariants, and F-jumping numbers. Furthermore, their test ideals coincide for $\lambda \in (0,1)$.

Remark 3.16. Let $f = x^a + y^b$ be a quasi-homogeneous plane curve, or a one-monomial μ-constant deformation $g = x^a + y^b + tx^\alpha y^\beta$, $t \in \mathbb{Z}$, defined over $\mathbb{C}$. Consider the sets $\mathcal{X}_f = \{p \mid p \equiv 1 \pmod{ab},\ p \text{ prime}\}$, $\mathcal{X}_g = \{p \mid p \equiv 1 \pmod{ab},\ p > a\beta + b\alpha - ab,\ p \text{ prime}\}$ as subspaces of $\operatorname{Spec}\mathbb{Z}$, which are infinite by Dirichlet's theorem on arithmetic progressions. In the Zariski topology on $\mathbb{Z}$, a nonempty open subset is the complement of the union of finitely many points, and hence must intersects both $\mathcal{X}_f$ and $\mathcal{X}_g$. It follows that these sets are dense in $\operatorname{Spec}\mathbb{Z}$.

Choose a prime $p \in \mathcal{X}_f$, and denote by f_p the reduction modulo p of f along $\mathbb{Z}[x,y] \to \mathbb{F}_p[x,y]$. Similarly, let g_p be the reduction of g, with $p \in \mathcal{X}_g$. The multiplier ideals $\mathcal{J}(f^\lambda)$, $\mathcal{J}(g^\lambda)$ are generated by polynomials over $\mathbb{Z}$, and can be obtained with an algorithm proposed by Blanco and Dachs-Cadefau [2]. After computing their reductions $\mathcal{J}(f^\lambda)_p$, $\mathcal{J}(g^\lambda)_p$, one sees they coincide with the test ideals, namely $\mathcal{J}(f^\lambda)_p = \tau(f_p^\lambda)$, and $\mathcal{J}(g^\lambda)_p = \tau(g_p^\lambda)$, for all λ. In consequence, the weak ordinarity conjecture (Conjecture 2.15) holds for quasi-homogeneous plane curves and their one-monomial μ-constant deformations.

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