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Existence of non-convex rotating vortex patches to the 2D Euler equation

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Resum *(CAT)*
Els únics exemples explícits de vòrtexs en rotació uniforme a les equacions d'Euler 2D són els cercles i les el·lipses. Els altres exemples dels quals es coneixen propietats quantitatives són propers a aquests.
En aquest article presentem l'existència de vòrtexs no convexos, lluny dels règims pertorbatius, podent obtenir-ne una descripció quantitativa precisa. Per demostrar-ho utilitzem una combinació d'anàlisi i tècniques de demostració assistida per ordinador.

Abstract *(ENG)*
The only explicit examples of uniformly rotating vortex patches to the 2D Euler equations are circles and ellipses. The other examples for which quantitative properties are known are close to these ones.
In this paper we present the existence of non-convex ones, far from the perturbative regimes, being able to obtain a precise quantitative description. To prove it we use a combination of analysis and computer assisted proofs techniques.

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1. Introduction

The behavior of ideal, incompressible fluids is governed by the Euler equations, a set of partial differential equations derived by Leonhard Euler in the 1750s. In two dimensions, these equations can be formulated in terms of vorticity, ω, which measures the local rotation of the fluid in the following way:

$$\begin{cases} \partial_t \omega + (K * \omega) \cdot \nabla \omega = 0 & \text{in } \mathbb{R}^2 \times [0, \infty), \\ \omega(\cdot, 0) = \omega_0(\cdot) & \text{in } \mathbb{R}^2, \end{cases}$$

where $K(\mathbf{y}) = \frac{\mathbf{y}^\perp}{2\pi|\mathbf{y}|^2}$. A special class of solutions, known as *vortex patches*, occurs when the initial vorticity is constant within a defined region D_0 and zero elsewhere. The vorticity remains constant along particle trajectories, so the patch simply deforms over time. A more exhaustive introduction with more detailed computations can be found in [10].

This study focuses on a particular type of vortex patch called a *V-state*, which is a patch that rotates with a uniform angular velocity (Ω) without changing its shape. Mathematically, if a patch is defined by a domain D_0 at time $t = 0$, its evolution is given by $D(t) = M(\Omega t)D_0$, where M is a rotation matrix.

The boundary of the patch ∂D_0, parametrized in polar coordinates by the function $R(\alpha)$, must satisfy the following non-local integro-differential equation:

$$R(\alpha)R'(\alpha) = F[R], \tag{1}$$

where $F[R]$ is an integral operator given by:

$$\begin{aligned} F[R] := {} & \frac{1}{4\pi\Omega}\int_0^{2\pi} \cos(\alpha-\beta)\log\left((R(\alpha)-R(\beta))^2+4R(\alpha)R(\beta)\sin^2\left(\frac{\alpha-\beta}{2}\right)\right)(R(\alpha)R'(\beta)-R'(\alpha)R(\beta))\,d\beta \\ & + \frac{1}{4\pi\Omega}\int_0^{2\pi} \sin(\alpha-\beta)\log\left((R(\alpha)-R(\beta))^2+4R(\alpha)R(\beta)\sin^2\left(\frac{\alpha-\beta}{2}\right)\right)(R(\alpha)R(\beta)+R'(\alpha)R'(\beta))\,d\beta. \end{aligned}$$

1.1. Previous work

The circle is a trivial solution, and in 1874, Kirchhoff ([9]) proved that ellipses are also V-states. Apart from these, no other explicit solutions are known. In the 1970s and 80s, numerical studies by Deem–Zabusky [3] and others revealed families of V-states with m-fold symmetry bifurcating from the circle, suggesting a rich variety of solutions beyond the classical examples.

Theoretical progress followed, with Burbea [1] and Hmidi, Mateu, and Verdera [6] proving the existence of local bifurcation branches from the disk for every integer symmetry $m \geq 3$. More recently, global bifurcation curves were constructed by Hassainia, Masmoudi, and Wheeler [5]. However, these powerful theoretical tools provide existence but lack quantitative information about the solutions far from the initial bifurcation point. This is a critical limitation because proving properties like non-convexity requires precise knowledge of the solution's shape. A general argument for all m-fold branches will fail, as the branch for $m = 2$ is known to be convex and for $m = 3$ it is expected.

This paper addresses this gap by providing the first positive existence result with quantitative information for a V-state far from the perturbative regions around the circle or ellipses. We prove the existence of a non-convex, 6-fold symmetric V-state.

Due to length constraints, most proofs are omitted from this article. A more thorough explanation of the results is detailed by Gómez-Serrano and the present author in [2].

1.2. Results

Theorem 1.1 (Main theorem)**.** *There exists an analytic solution $R(x)$ of the V-state equation* (1)*, with* $\Omega = \frac{1537}{3750}$*, such that its associated vortex patch $D \subset \mathbb{R}^2$ is non-convex and has* 6*-fold symmetry. See Figure 1.*

Corollary 1.2. *There exists $\delta > 0$ such that for any angular velocity in $(\Omega - \delta, \Omega + \delta)$ there exists an analytic solution R to* (1) *with* 6*-fold symmetry, where Ω is the angular velocity given by Theorem 1.1.*

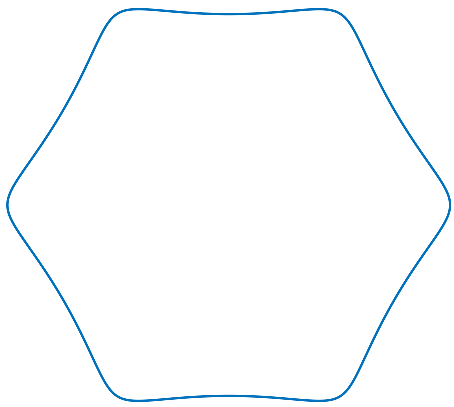

Figure 1: The boundary ∂D is a curve contained in the plotted line.

Proof of Theorem 1.1. In Section 2 we prove the existence of a solution close to R_0, in Section 3 we prove the analyticity and non-convexity and in the last Section 4 we explain the computer assisted parts of the proof. □

2. Existence

Our approach is to formulate the problem as a fixed point equation and solve it using a combination of analytical estimates and rigorous numerical computations. The core idea is to find an approximate solution and then prove that a true solution exists in its vicinity.

2.1. Fixed point equation around the approximate solution

The first step is to find a highly accurate approximate solution, $R_0(x)$. We define it as a truncated Fourier series with 6-fold symmetry ($m = 6$):

$$R_0(x) := \sum_{k=0}^{N_0} c_k \cos(kmx)$$

with $N_0 = 30$. The coefficients c_k and the angular velocity Ω are chosen to make the error of this approximation, $E[0](x) = R_0'(x)R_0(x) - F[R_0](x)$, as small as possible. We seek a true solution $R(x)$ as a perturbation of $R_0(x)$:

$$R(x) = R_0(x) + v(x) \quad \text{where} \quad v(x) = \int_0^x \tilde{u}(y)\, dy. \tag{2}$$

Here, u is a function in $L^2([0, \pi/m])$ and $\tilde{u}$ is its odd, $2\pi/m$ periodic extension. This formulation is designed to fix the scaling and rotation symmetries of the problem while preserving the m-fold symmetry.

Taking the Fréchet derivative of the equation (1), we can write the equation for the perturbation as $\mathcal{L}v = E[0] + \mathcal{N}_L[v]$, where $\mathcal{L}$ is a linear operator and $\mathcal{N}_L[v] = O(v^2)$ is nonlinear. Using the expression (2) and the symmetries of $\tilde{u}$, we write this equation in terms of u in the following way:

$$Lu(x) = E[0](x) + N_L[u](x) \quad \forall x \in [0, \pi/m].$$

- L is a linear operator defined as $Lu := u + \int_0^{\pi/m} K(x, y)u(y)\, dy$, where K is a $L^2\left([0, \frac{\pi}{m}]^2\right)$ function that depends on the approximate solution R_0 in a nontrivial way.
- E_0 is the error of the approximate solution R_0 as we have mentioned before.
- N_L is a nonlinear operator that contains higher-order terms of the perturbation u.

If we prove that L is invertible, the problem is now reduced to showing that the operator $Gu := L^{-1}(E[0] + N_L[u])$ has a fixed point in a small ball around the origin in an appropriate function space. We choose the space $L^2([0, \pi/m])$ and seek a solution u within a ball of radius $\epsilon = 3 \cdot 10^{-5}$.

2.2. Inverting the linear operator

A direct inversion of the operator L is not feasible, since it depends very nonlinearly (and non-locally) on R_0. Instead, we use a computer assisted approach. The main idea to prove the invertibility of L is to first approximate L by $L_F =$ Identity $+$ Finite Rank, then prove the invertibility of L_F and finally prove that the approximation error $L - L_F$ is small enough, making L invertible via a Neumann series.

Definition 2.1. Let $\{e_n(x)\}_n$ be the normalized Fourier basis of $X_m = L^2\left([0, \frac{\pi}{m}]\right)$. Also let $N = 201$, and $E_N = \text{span}\{e_n\}_{n=1}^N$ be the subspace generated by the first N vectors. Similarly, let $E_N^\perp$ be its orthogonal subspace. We will also define $L_F = I + \mathcal{K}_F \colon X_m \mapsto X_m$ where $\mathcal{K}_F \colon E_N \to E_N$ is given by

$$\mathcal{K}_F[u] = \int_0^{\frac{\pi}{m}} K_F(x, y)u(y)\, dy \quad \text{with} \quad K_F(x, y) := \sum_{k,l=1}^N A_{k,l}e_k(x)e_l(y),$$

where $A_{k,l}$ is finite explicit matrix very close to the projection of the operator $K(x, y)$ to the E_N subspace.

The proof of the next lemma is computer assisted and will be explained in Section 4.

Lemma 2.2. *The matrix $I + A$ is invertible and satisfies $\|(I + A)^{-1}\|_2 \le C_2$, with $C_2 := 8.8$.*

Lemma 2.3. *The operator L_F is invertible and $\|L_F^{-1}\|_2 < C_2$ with $C_2 := 8.8$.*

Proof. Let P_N be the projection operator onto E_N. Using that $\mathcal{K}_F = P_N\mathcal{K}_F = \mathcal{K}_F P_N$, L_F decouples in $E_N \bigoplus E_N^{\perp}$ as

$$L_F = \begin{pmatrix} I_{E_N} + A & \mathbf{0} \\ \mathbf{0} & I_{E_N}^{\perp} \end{pmatrix};$$

then as E_N is a finite dimensional vector space, to invert $I_{E_N} + A$ we have to invert the corresponding matrix, and the identity in $E_N^{\perp}$ is trivially inverted, so

$$L_F^{-1} f := (I_{E_N} + A)^{-1} P_N f + (I - P_N) f.$$

We can conclude that L_F is invertible. Moreover

$$\begin{aligned}\|L_F^{-1} f\|_{L^2} = \|P_N L_F^{-1} f\|_{L^2} + \|(I - P_N) L_F^{-1} f\|_{L^2} &\leq \|(I_{E_N} + A)^{-1}\|_2 \|P_N f\|_{L^2} + \|(I - P_N) f\|_{L^2} \\ &\leq \max\{\|(I_{E_N} + A)^{-1}\|_2, 1\} \|f\|_{L^2},\end{aligned}$$

hence its norm is bounded by C_2 because by Lemma 2.2, $\|(I + A)^{-1}\|_2 \leq C_2$ and $1 < C_2$. □

The proof of the next lemma is again computer assisted and it will be explained in Section 4.

Lemma 2.4. *The error of approximating the operator* L *by* L_F *satisfies* $\|L - L_F\|_2 \leq C_3$, *with* $C_3 := 0.085$.

We can now state and prove the main result of this subsection.

Proposition 2.5. *The linear operator* $L\colon X_m \to X_m$ *is invertible. Moreover* $\|L^{-1}\|_2 \leq C_1 = 35$.

Proof. Using that L_F is invertible, we can write

$$L = L_F (I + L_F^{-1}(L - L_F)).$$

We can then invert $I + L_F^{-1}(L - L_F)$ using a Neumann series because due to Lemmas 2.3, 2.4 we have that $\|L_F^{-1}(L - L_F)\|_2 \leq C_2 C_3 < 1$. As L_F is also invertible, we can conclude that L is invertible and

$$\|L^{-1}\|_2 = \|(I + L_F^{-1}(L - L_F))^{-1} L_F^{-1}\|_2 \leq \frac{C_2}{1 - C_2 C_3} = \frac{8.8}{0.252} < 35 = C_1.$$ □

2.3. Solving the fixed point equation

The goal is to use Banach Fixed Point theorem to prove the existence of solutions. For this we need control over the Lipschitz norm of G. We are going to state a proposition that together with the estimates on L^{-1} is going to allow us to prove the existence of a fixed point.

Proposition 2.6. *We have the following bounds on the nonlinear terms:*

(i) $\|E[0]\|_{L^2} \leq \epsilon_0$,

(ii) $\|N_L[u]\|_{L^2} \leq C_5 \|u\|_{L^2}^2$,

(iii) $\|N_L[u_1] - N_L[u_2]\|_{L^2} \leq \epsilon C_6 \|u_1 - u_2\|_{L^2}$

for $\epsilon_0 := 3 \cdot 10^{-7}$ *and some explicit positive constants* C_5, C_6.

The proof of the bound on the error of the approximate solution, the first bound of last lemma, is computer assisted. The rest of them are estimated by hand.

The next proposition is the main one of this section.

Proposition 2.7. *The operator $G[u] := L^{-1}(E[0] + N_L[u])$ has a fixed point in the ball of radius ϵ.*

Proof. We check that the operator maps the ball $B_\epsilon(0)$ into itself:

$$\|G[u]\|_{L^2} \leq C_1(\epsilon_0 + C_5\epsilon^2) < \epsilon,$$

and that it is Lipschitz

$$\|G[u_1] - G[u_2]\|_{L^2} \leq C_1 C_6 \epsilon \|u_1 - u_2\|_{L^2} < \|u_1 - u_2\|_{L^2}.$$

In both inequalities we have checked that the explicit values of the constants satisfy them. We conclude by Banach Fixed Point theorem. □

Once we have proven the existence of this solution it is possible to check that indeed $R = R_0 + \int_0^x \tilde{u}$ satisfies equation (1).

3. Non-convexity and improved regularity

3.1. Proof of non-convexity

To prove this, we first note that using Hölder inequality and the bound $\|u\|_{L^2} \leq \epsilon$ we get the following enclosure for the solution $R(x)$

$$R_{\inf} := R_0(x) - \sqrt{\frac{\pi}{m}|\sin(mx)|} \leq R(x) \leq R_0(x) - \sqrt{\frac{\pi}{m}|\sin(mx)|} =: R_{\sup}(x).$$

The quantitative control we have on the solution allows us to rigorously prove its non-convexity. Let's define P_0, P_1 the points on the boundary of the patch at angle 0, $2\pi/m$. We will then prove that $R(\pi/m) < \left|\frac{P_0+P_1}{2}\right|$ so the midpoint does not belong to the domain of the patch D.

$$R(\pi/m) \leq R_{\sup}(\pi/m) = R_0\left(\frac{\pi}{m}\right) + \epsilon\sqrt{\frac{\pi}{m}} < R_0(0)\cos\left(\frac{\pi}{m}\right) = \left|\frac{P_0 + P_1}{2}\right|,$$

where the explicit inequality in the middle is checked using interval arithmetic with the computer.

3.2. Further regularity

We initially proved that R is an H^1 function. However, we can show that it is much smoother.

1. C^∞ **Regularity**: We use a bootstrapping argument. By rewriting the V-state equation, we show that if the k-th derivative $\partial^k R$ is bounded, then $\partial^{k+1} R$ is also bounded, and by induction, R is C^∞. The argument involves carefully differentiating the non-local equation and controlling the singularities in the integral kernels to obtain the following inequality:

$$\|\partial^{k+2}R\|_{L^\infty} \lesssim 1 + \delta(1 + \|\partial^{k+1}R\|_{L^\infty}^{\beta_k})\|\partial^{k+2}R\|_{L^\infty} + \frac{1}{\delta}\|\partial^{k+1}R\|_{L^\infty}^{\gamma_k},$$

for a sufficiently small $\delta > 0$.

2. **Analyticity**: To prove that the boundary is analytic, we rephrase the problem as a free boundary elliptic problem for the stream function ψ. Then we can explicitly check that the gradient is not vanishing in the boundary. Once we have this the analyticity is a direct consequence due to [8, Theorem 3.1'].

4. Computer assisted estimates

The proof of the main theorem has strongly relied in using the computer to perform difficult computations. We will explain in which steps and how we were able to obtain the required bounds. We refer to [4] for a thorough overview of computer assisted proofs in PDEs.

To perform rigorous computations with a computer, we cannot use floating-point arithmetic, as the results would not be rigorous because we have no control over the rounding error. Therefore, we must use interval arithmetic. Interval arithmetic treats numbers as intervals, where the interval bounds are numbers representable with a fixed number of digits. In this arithmetic, operations are defined between intervals in such a way that for all numbers belonging to an input interval, the result of the operation is guaranteed to be included in the final output interval. This method allows for the propagation of errors to ultimately obtain an interval that contains the exact result.

We used the `C++` library `Arb` for these computations [7]. The key computer assisted steps were:

- **Bounding the matrix inverse**: Finding a rigorous upper bound for $\|(I+A)^{-1}\|_2$ by computing a lower bound for the minimum eigenvalue of the symmetric matrix $(I+A)^T(I+A)$.

- **Bounding integrals**: Using high-order quadrature rules with rigorous error bounds to compute norms of functions involving difficult integrals, such as the error of the approximate solution $\|E[0]\|_{L^2}$ and the approximation error of the linear operator $\|K-K_F\|_{L^1_x L^\infty_y}, \|K-K_F\|_{L^1_y L^\infty_x}$. Special care was taken to handle the logarithmic singularities in the integrands.

- **Verifying final inequalities**: Checking the conditions for the Banach Fixed Point theorem and the final non-convexity inequality by plugging in the rigorously computed bounds for all constants.

The numerics required a nontrivial amount of computing size; for instance, bounding the error of the approximate solution to the order of 10^{-7} took about 9 hours in 64 CPUs, and the bounding of the error of the kernel approximation took 29 hours in also 64 CPUs.

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