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Monodromy conjecture for Newton non-degenerate hypersurfaces

Oriol Baeza Guasch
Universitat Politècnica de Catalunya
oriol.baeza@estudiantat.upc.edu

Resum *(CAT)*
Aquest treball estudia la Conjectura Forta de la Monodromia (SMC) en la versió topològica. Després d'introduir els conceptes de resolució de singularitats, polinomi de Bernstein–Sato i la funció zeta, esbocem els resultats involucrats en la demostració de la SMC per a singularitats Newton no degenerades (NND). Aquesta prova requereix, però, hipòtesis addicionals sobre els nombres del residu, i construïm exemples que mostren que no poden ometre's, la qual cosa suggereix que calen altres tècniques per a atacar el cas general.

Abstract *(ENG)*
This work studies the Strong Monodromy Conjecture (SMC) in its topological setting. After introducing the concepts of resolution of singularities, Bernstein–Sato polynomial, and the zeta function, we sketch the results involved in the proof of the SMC for Newton non-degenerate (NND) singularities. This approach requires nonetheless additional hypothesis on the residue numbers, and we construct examples showing that they can't be dropped, which suggests that new techniques are needed to attack the general case.

Keywords: *monodromy, Bernstein–Sato polynomial, resolution of singularities, plane curves, Newton non-degenerate.*
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1. Introduction

The monodromy conjecture is a problem in the field of singularity theory in algebraic geometry, formulated by the Japanese mathematician Igusa in the seventies, which relates two invariants of a singularity. On the one hand, the roots of a polynomial arising from a functional equation satisfied by the singularity (the so called Bernstein–Sato polynomial). On the other hand, the poles of the zeta function (in our setting, the topological version), which contains information about a resolution of the singularity. More precisely, the conjecture predicts that every pole of this zeta function is a root of the Bernstein–Sato.

Although the general case remains open, a positive result has been proven for some special cases. In particular, it is known to be true for plane curves (Loeser '88), for Newton non-degenerate (NND) polynomials modulo an hypothesis on the so called *residue numbers* (Loeser '90), as well as in certain hyperplane arrangements, or also semi-quasihomogeneous singularities.

Both in the cases of plane curves and NND polynomials, a more combinatorial approach is possible, which simplifies some computations and allows to use some technical cohomological results. Nonetheless, for the NND case, this comes with the price of adding two hypothesis on the residue numbers. We discuss the possibility of removing the hypothesis, and show that they do not hold in general. Even more, we will see that divisors not satisfying them can still contribute to the poles of the topological zeta function, suggesting that this approach won't work for the general case.

2. Preliminaries

2.1. Complex zeta function and resolution of singularities

Before stating the conjecture, we must introduce the two main objects of the problem: the Bernstein–Sato polynomial and the topological zeta function. Even more, to give some context and motivation of the statement, we must first begin with the complex zeta function.

The complex zeta function, for a polynomial f and a test function ϕ (meaning a complex function $\mathbb{C}^\infty$ with compact support), is defined as

$$Z(s) = Z(f, \phi; s) := \int_{\mathbb{R}^n} |f(x)|^s \phi(x)\, \mathrm{d}x,$$

where technically we must understand this a distribution in the space of test functions. It can be checked that $Z(s)$ converges and is holomorphic in the semiplane $\Re(s) > 0$. Its meromorphic continuation and the distribution of the possible poles was posed as a problem by I. Gelfand [11, §3.I], and solved in two different manners.

On one hand, we can use a resolution of singularities (guaranteed in characteristic 0 by Hironaka [12]). Recall that an embedded resolution of a polynomial f is a proper morphism $\pi \colon Y \to X$ such that Y is smooth, the restriction of π outside the singular locus is an isomorphism, and that around each point in the preimage we have a neighborhood and a chart over which $\pi^* f = u(y) y_{i_1}^{N_1} \cdots y_{i_r}^{N_r}$ with $u(0) \neq 0$ a unit and $N_i \geq 0$ integers. From the local expression, we can write the pullback divisor globally as

$$\operatorname{div}(\pi^* f) = \sum_{j \in J} N_j E_j,$$

where $(E_i)_{i\in J}$ are the irreducible components of $\pi^{-1}(f^{-1}(0))$, each E_i given in local coordinates by $\{x_i = 0\}$, respectively. Another relevant numerical quantity that will appear are the coefficients in the global expression of the pullback of the standard volume form

$$\operatorname{div}(\pi^*(\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n)) = \sum_{j\in J}(k_j - 1)E_j.$$

In this context, we can use the resolution as a change of variables in the integral and deduce that the poles of the complex zeta function are of the form $-\frac{k_j+\nu}{N_j}$, with ν a non-negative integer, and (k_j, N_j) the numerical data associated to the exceptional divisors, introduced above.

2.2. Bernstein–Sato polynomial

On the other hand, we can introduce the Bernstein–Sato polynomial, which is an analytical invariant (and not a topological one) of the singularity. First, consider $R := \mathbb{C}[x_1, \dots, x_n]$ (or more generally the ring of holomorphic functions or even formal power series) and denote $\mathscr{D} := R\langle\partial_1, \dots, \partial_n\rangle$ the Weyl algebra. All elements commute except for the relations $\partial_i x_i - x_i\partial_i = 1$, and so it is easy to show that any element (a priori only a linear differential operator) can be written as a finite sum $P = \sum_{\alpha,\beta} a_{\alpha,\beta}x^\alpha\partial^\beta$. For a more gentle introduction and more details on the properties of the Weyl algebra, we refer to [5]. Next, consider the polynomial ring $\mathscr{D}[s] := \mathscr{D} \otimes_{\mathbb{C}} \mathbb{C}[s]$ with new variable s, and note that the free module $R_f[s]\cdot f^s$ has a natural structure of left $\mathscr{D}[s]$-module given by the product rule. Indeed, every element of the module can be written as $\frac{g}{f^k}\cdot f^s$ for some $g(x,s) \in R[s]$, and the action of the partial derivatives is

$$\partial_i \cdot \left(\frac{g}{f^k}\cdot f^s\right) = \partial_i \cdot \left(\frac{g}{f^k}\right)\cdot f^s + \frac{sg}{f^{k+1}}\cdot\frac{\partial f}{\partial x_i}\cdot f^s.$$

The relevant result is then the existence of solutions to the following functional equation, which was first proven by Bernstein [2] for the case of polynomials, and later by Kashiwara [13] and Björk [3] in the cases of holomorphic functions and formal series, respectively.

Theorem 2.1 ([2])**.** *Let $f \in R$ be a polynomial. Then, there exists a polynomial $P(s) \in \mathscr{D}[s]$ and a polynomial $b_{f,P}(s) \in \mathbb{C}[s]$ such that the relation*

$$P(s)f^{s+1} = b_{f,P}(s)f^s \tag{1}$$

holds formally in the $\mathscr{D}$-module $R_f[s]\cdot f^s$.

The set of polynomials $b_{f,P}$ satisfying such a differential equation as above form an ideal in $\mathbb{C}[s]$, so we can consider its monic generator: the *Bernstein–Sato polynomial* of f denoted by $b_f(s)$.

Example 2.2. For $f = x_1^2 + \cdots + x_n^2$ we have $b_f(s) = (s+1)(s+n/2)$, as taking $P(s)$ to be the Laplacian operator (thought as a constant polynomial in $\mathscr{D}[s]$), we have the relation

$$\left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right) f^{s+1} = 4(s+1)\left(s+\frac{n}{2}\right) f^s.$$

Example 2.3. For $f = x^2 + y^3$ in $\mathbb{C}[x, y]$ we have the following relation

$$\left[\frac{1}{12}\frac{\partial}{\partial x}\frac{\partial}{\partial y}y\frac{\partial}{\partial x} + \frac{1}{27}\left(\frac{\partial}{\partial y}\right)^3 + \frac{1}{4}\left(s+\frac{7}{6}\right)\left(\frac{\partial}{\partial x}\right)\right] f^{s+1} = (s+1)\left(s+\frac{5}{6}\right)\left(s+\frac{7}{6}\right) f^s,$$

and it can be proved that $b_f(s) = (s+1)(s+\frac{5}{6})(s+\frac{7}{6})$.

Now, having introduced the functional equation (1), the idea is to use it to integrate by parts and obtain a meromorphic continuation of $Z(s)$ to the whole complex plane, as for any $r \in \mathbb{N}$ we have

$$Z(s) = \frac{1}{b_f(s+r-1)\cdots b_f(s+1)b_f(s)} \int_{\mathbb{R}^n} f(x)^{s+r}\phi_r(x)\,\mathrm{d}x, \quad \Re(s) > -r.$$

In this case, it can be seen that the poles of the zeta function are of the form $\lambda - \nu$ for λ a root of $b_f(s)$ and ν a non-negative integer. By comparing this with the previous candidate quantities for the poles, one arrives at the following result.

Theorem 2.4. *Every root of the Bernstein–Sato polynomial b_f is of the form $-\frac{k_j+\nu}{N_j}$ for some $j \in J$ and ν a non-negative integer.*

2.3. Monodromy conjecture

Altogether, the relation between poles of the complex zeta function and the roots of the Bernstein–Sato is clear. Motivated by this, and after computing some examples, Igusa formulated the conjecture for the poles of the p-adic zeta function, and later stated in the topological and motivic settings too.

In the topological version, we have left to introduce the topological zeta function, first defined by Denef and Loeser in [8], who formalized its definition from a heuristic argument taking a limit of the p-adic zeta function, and showing that the following expression is independent of the choice of a resolution (there is currently no known intrinsic definition).

Definition 2.5 (Topological zeta function)**.** Let $f \in \mathbb{C}[x_1, \dots, x_n]$ be a non-constant polynomial, and choose a resolution $\pi\colon Y \to \mathbb{A}^n_{\mathbb{C}}$ of $\{f = 0\}$. The (global) *topological zeta function* of f is

$$Z_{\mathrm{top}}(f; s) := \sum_{I \subset J} \chi(E_I^\circ) \prod_{i \in I} \frac{1}{k_i + N_i s}, \quad E_I^\circ = \bigcup_{j \in I} E_j \setminus \bigcup_{j \notin I} E_j, \text{ for } I \subset J.$$

Then, it is clear from the expressions that the poles are still related with the roots of the Bernstein–Sato, as they both contain information from a resolution. However, the difficulty lies, on one side, in determining which poles remain in the zeta function after possible cancellation, and in the other, what candidate values in Theorem 2.4 are actually roots. The monodromy conjecture partly answers this, by predicting that every pole of the topological zeta function is a root.

Conjecture 2.6 (Topological monodromy conjecture)**.** *Let $f \in \mathbb{C}[x_1, \dots, x_n]$ be a non-constant polynomial. If s_0 is a pole of $Z_{\mathrm{top}}(f; s)$, then*

(i) (standard) $e^{2\pi\imath\Re(s_0)}$ *is an eigenvalue of the monodromy of $f\colon \mathbb{C}^n \to \mathbb{C}$ at a point of $\{f = 0\}$.*

(ii) (strong) *s_0 is a root of the Bernstein–Sato polynomial $b_f(s)$.*

In this work, we always refer to the strong version, which implies the standard (or weak) one thanks to a result by Malgrange ([20, Proposition 7.1]). So far, the approaches for the solved cases do not provide a clear conceptual idea why the result should be true, apart from the analogy with the complex zeta function (see [22, Remark 2.13]).

A key common element in the proofs of the known cases is the study of periods of integrals (see [18, 19]). These objects allow to relate its asymptotic behavior with the monodromy action (that is, the action of a generator of the fundamental group of a punctured disk around the singularity of the homology groups of a fiber X_t of the Milnor fiber). Even more, in this context Malgrange is able to prove that certain quantities appearing in the asymptotic expressions are roots of $b_f(s)$.

Nonetheless, this approach requires the existence of a non-zero cohomology class in $H_n(X_t, \mathbb{C})$. This result is precisely what Deligne and Mostow prove in the case of plane curves ([7, Proposition 2.14]), and Loeser for the case of Newton non-degenerate ([15, Théorème 3.7]) adapting a result by Esnault and Viehweg ([9]).

Furthermore, in the case of plane curves, the cohomological result of existence only applies to rupture divisors of the resolution. For this reason, Loeser complements it with a combinatorial study of a (minimal) resolution of the singularity, represented in the so called *dual graph*. In this graph, each vertex represents a rupture divisor E_i, $i \in J$, of the resolution, and the edges are added and modified after each *blow up*. In this situation, it is possible to study the quotient k_i/N_i and the residue numbers $\varepsilon(i,j) = k_j - N_j k_i/N_i$ over the divisors via recurrences in the dual graph. In particular, it can be proven that the only possible poles contributing to the topological zeta function arise from the rupture divisors, and so the aforementioned cohomological result is enough.

As for the NND case, in the next sections we will see how the technical hypothesis required for the cohomological result lead to additional conditions on the residue numbers. We will show that the relevant hypothesis is the second one, and we will construct examples that prove that they are not always satisfied. Even worse, we will see that the *bad* divisors violating it can contribute with non-zero residue to a pole of the topological zeta function.

3. Newton non-degenerate polynomials

3.1. Definition and properties

Polynomials which are Newton non-degenerate are sometimes also called non-degenerate with respect to the Newton polytope, or simply non-degenerate. The condition that these polynomials satisfy has a more combinatorial flavor, as it is easier described by considering the objects that we will introduce next.

We consider a polynomial $f(x_1, \ldots, x_n) = \sum_{p \in \mathbb{N}^n} a_p x_1^p \cdots x_n^p$ such that $f(0) = 0$. For brevity, we will use multi-index notation when convenient $f(x) = \sum_{p \in \mathbb{N}^n} a_p x^p$, and we define its support to be $\operatorname{supp}(f) = \{p \in \mathbb{N}^n \mid a_p \neq 0\}$.

Definition 3.1 (Newton polyhedron)**.** Let $f = \sum_{p \in \mathbb{N}^n} a_p x^p \in \mathbb{C}[x]$ with $f(0) = 0$. We define the *global Newton polyhedron* $\Gamma_{gl}(f)$ of f as the convex hull of $\operatorname{supp}(f)$. Also, we define the *local Newton polyhedron* $\Gamma(f)$ as the convex hull of the set $\bigcup_{p \in \operatorname{supp}(f)} p + (\mathbb{R}_{\geq 0})^n$.

We will use the term *face* of $\Gamma(f)$ to refer to any convex subset τ that can be obtained by intersecting the Newton diagram with a hyperplane H of $\mathbb{R}^n$ such that $\Gamma(f)$ is contained in one of the half-spaces defined by H. Note that we also consider the total polyhedron as a face.

Definition 3.2 (Non-degenerate)**.** We say that f is Newton non-degenerate at 0 if for any face $\tau \subset \Gamma(f)$, the hypersurface defined by the truncation $f^\tau := \sum_{p \in \tau \cap \mathrm{supp}(f)} a_p x^p$ satisfies that the polynomials $x_i \frac{\partial f^\tau}{\partial x_i}$ for $i = 1, \dots, n$ do not vanish at the same time in $(\mathbb{C} \setminus 0)^n$.

This class of NND polynomials is general enough to be of interest, while also allows a more combinatorial treatment of the problem. We will now see how the polyhedron and its dual fan encode the information of a good resolution of the singularity. For that, we first introduce some notation.

Definition 3.3 (N, k, F)**.** For a vector $a \in (\mathbb{R}^+)^n$, we define the quantities $N(a) := \inf_{x \in \Gamma(f)}\{\langle a, x\rangle\}$ and $k(a) := \sum_{i=1}^n a_i$. Also, define the first meet locus $F(a) := \{x \in \Gamma(f) \mid \langle a, x\rangle = N(a)\}$, which is a proper face of $\Gamma(f)$ if $a \neq 0$, and $F(0)$ recovers the whole diagram.

Definition 3.4 (Dual fan)**.** For τ a face of $\Gamma(f)$, we define the *cone associated* to τ as

$$\Delta_\tau := \{a \in (\mathbb{R}_{>0})^n \mid F(a) = \tau\}/\sim, \quad a \sim a' \text{ iff } F(a) = F(a').$$

The collection of these cones for all faces of the Newton polytope as the *dual fan*.

As an example, see the following Figure 1, where we consider the plane curve defined by $f = x^3 - y^2 + 4xy + 3x^2y$, and construct the Newton polyhedron with labeled faces and corresponding truncations (left), and the associated dual fan (right).

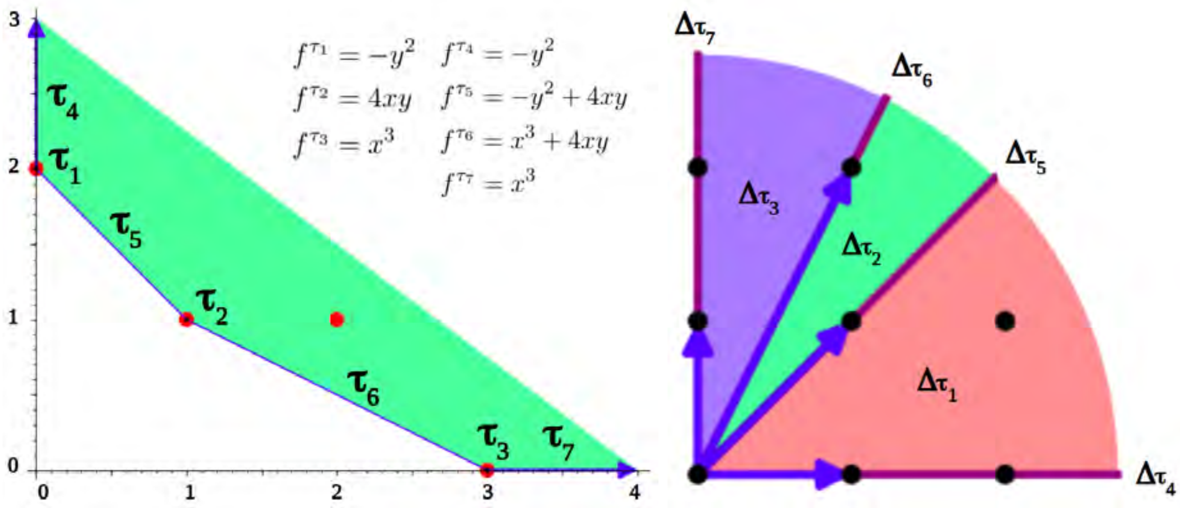

Figure 1: $\Gamma(f)$ and dual fan of the polynomial $f = x^3 - y^2 + 4xy + 3x^2y$.

Next, we recall the following properties of cones.

Definition 3.5 (Cone)**.** A *convex polyhedral cone*, or *cone* for short, is a set

$$C = \{\lambda_1 v_1 + \dots + \lambda_s v_r \in V \mid \lambda_i \geq 0\} = \mathbb{R}_{\geq 0} v_1 + \dots + \mathbb{R}_{\geq 0} v_r,$$

where V is an n-dimensional vector space over $\mathbb{R}$, and the vectors $\{v_i\}$ are called the *generators* of the cone. The dimension of C is defined to be the dimension of the smallest vector space containing it.

We say that the cone is *simplicial* if its generating vectors are linearly independent over $\mathbb{R}$. Moreover, we will say it is *simplicial rational* if on top of that the entries of the vectors are integers. We say that the cone is *regular* (or *simple*) if the set of generating vectors can be extended to a base of the $\mathbb{Z}$-module $\mathbb{Z}^n$.

Then, the key theorem from toric geometry that we will need is the following.

Theorem 3.6 ([14, pp. 32–25]). *Let Δ be a cone generated by vectors $v_1, \dots, v_r \in \mathbb{R}^n \setminus \{0\}$. There exists a finite partition of Δ in cones Δ_i, such that each cone is generated by a subset of linearly independent vectors of $\{v_1, \dots, v_r\}$. Moreover, if Δ is simplicial rational, a partition in regular cones can be obtained by introducing suitable new generating rays.*

Such a regular simplicial subdivision can be obtained algorithmically (see also [1, §8.2.2]), and in particular applying it to all the cones in the dual fan, we deduce the next result.

Theorem 3.7 ([1, Lemma 8.7]). *There exists a fan consisting of regular simplicial cones which is obtained as a subdivision of the dual fan associated to the Newton polyhedron.*

Remark 3.8. Notice, however, that we haven't claimed anything about uniqueness, as there can exist multiple valid subdivisions in simplicial regular cones.

3.2. Resolution and topological zeta function from the dual fan

Back to resolution of singularities, we know thanks to the result by Hironaka that we always have one, and that it can be obtained as a composition of blowups. However, in this setup we will obtain a resolution more directly via a single toric blowup, which we define next (see a more detailed introduction in [21] and [17], and also [10, 6] for a complete treatment of toric varieties).

Definition 3.9 (Toric blowup). Consider a unimodular integral $n \times n$ matrix $\sigma = (\sigma_{i,j})$, and define the *toric blowup* (or *modification*) associated toσ as the birational morphism

$$\begin{aligned} \pi_\sigma \colon (\mathbb{C}^*)^n &\to (\mathbb{C}^*)^n \\ (x_1, \dots, x_n) &\mapsto (x_1^{\sigma_{1,1}} x_2^{\sigma_{1,2}} \cdots x_n^{\sigma_{1,n}}, \dots, x_1^{\sigma_{n,1}} x_2^{\sigma_{n,2}} \cdots x_n^{\sigma_{n,n}}). \end{aligned}$$

In particular, if we have a regular simplicial cone in Σ^* of maximum dimension given by vectors $\{r_1, \dots, r_n\}$, we can consider the unimodular matrix $\sigma = (r_1 \quad r_2 \quad \cdots \quad r_n)$ and associate to it the birational map $\pi_\sigma \colon \mathbb{C}^n_\sigma \to \mathbb{C}^n$, which we ought to think about as one of the different charts of the resolution. With that, we construct a non-singular variety X as the quotient of the disjoint union $\bigsqcup_\sigma \mathbb{C}_\sigma$ over all the regular cones with the following identification. Two points $x \in \mathbb{C}^n_\sigma$ and $y \in \mathbb{C}^n_\tau$ are identified if, and only if, the birational map $\pi_{\tau^{-1}\sigma}$ is defined at the point x and $\pi_{\tau^{-1}\sigma}(x) = y$.

It can be verified that X is non-singular, and the maps $\{\pi_\sigma \colon \mathbb{C}^n_\sigma \to \mathbb{C}^n \mid \sigma \text{ regular simplicial cone}\}$ glue into a proper analytic map $\pi \colon X \to \mathbb{C}^n$.

Definition 3.10 (Associated toric blowup). The map $\pi \colon X \to \mathbb{C}^n$ is called the *toric blowup* (or modification) *associated* with Σ^* at the origin, where Σ^* is a regular simplicial cone subdivision of Σ.

Finally, we arrive at the result justifying our claim that the geometric properties of the Newton polyhedron contains the information of a resolution for NND singularities.

Theorem 3.11 ([21, p. 101]). *If f is Newton non-degenerate, then the associated toric blowup $\pi \colon X \to \mathbb{C}^n$ is a good resolution of f as a germ at the origin.*

In the same combinatorial spirit, it is possible to obtain a more explicit expression for the topological zeta function. For that, let us first introduce the following terms.

Definition 3.12. Let τ be a face in $\Gamma(f)$, and consider a decomposition of the associated cone $\Delta_\tau = \bigcup_{i=1}^r \Delta_i$ in simplicial cones of dimension $\dim \Delta_\tau = l$ such that $\dim(\Delta_i \cap \Delta_j) < l$, for all $i \neq j$. Then, define

$$J(\tau, s) := \sum_{i=1}^{r} J_{\Delta_i}(s), \quad \text{with} \quad J_{\Delta_i}(s) = \frac{\operatorname{mult}(\Delta_i)}{(N(a_{i_1})s + k(a_{i_1})) \cdots (N(a_{i_l})s + k(a_{i_l}))}$$

being $a_{i_1}, \dots, a_{i_l} \in \mathbb{N}^n$ the linearly independent primitive integral vectors that generate Δ_i. Lastly, if $\tau = \Gamma(f)$, we rather take $J(\tau, s) = 1$.

Remark 3.13. By [8, Lemme 5.1.1], the definition of $J(\tau, s)$ is independent of the choice of the decomposition of Δ_τ in simplicial cones.

Theorem 3.14 ([8, Théorème 5.3])**.** *Let $f \in \mathbb{C}[x]$ be a polynomial Newton non-degenerate, then*

$$Z(s) = \sum_{\substack{\tau \text{ vertex} \\ \text{of } \Gamma(f)}} J(\tau, s) + \left(\frac{s}{s+1}\right) \sum_{\substack{\tau \text{ face of } \Gamma(f) \\ \dim \tau \geq 1}} (-1)^{\dim \tau} (\dim \tau)! \operatorname{Vol}(\tau) J(\tau, s).$$

As the poles of $Z(s)$ arise from the poles of the terms $J(\tau, s)$ in the sum, we see that they are still either -1 or of the form $-k(a)/N(a)$ (see also [16, Théorème 5.3.1]), which justifies the choice of notation.

3.3. Sketch of the proof and additional hypothesis

We are now ready to sketch the approach by Loeser in the proof of the monodromy conjecture for the Newton non-degenerate case. One relevant subtlety is, as stated in Remark 3.8, that the regular subdivision is not unique, and therefore so are the residue numbers. For that reason, Loeser decides to work with *toric* residue numbers computed directly from the original dual fan, without performing a regular subdivision.

Definition 3.15 (Toric residue numbers)**.** If τ, τ' are two distinct faces of codimension 1 of the Newton polyhedron at the origin of f, we denote by $\beta(\tau, \tau')$ the greatest common divisor of the minors of order 2 of the matrix $(a(\tau), a(\tau'))$, where $a(\tau)$ is a primitive integral vector defining the face τ. Additionally, set

$$\lambda(\tau, \tau') = k(\tau') - \frac{k(\tau)}{N(\tau)} N(\tau'), \quad \varepsilon(\tau, \tau') = \lambda(\tau, \tau')/\beta(\tau, \tau'),$$

whenever $N(\tau) \neq 0$, which is the case if τ is a compact face.

However, the drawback is that the resolution obtained from the original dual fan need not be minimal, and the subsequent computations require special care, leading to the introduction of this β factor which appears as the degree of a finite morphism between singular toric varieties.

Then, the positive result proven by Loeser is the following.

Theorem 3.16 ([16, Théorème 5.5.1])**.** *Let f be a comfortable polynomial verifying $f(0) = 0$, with Newton diagram $\Gamma(f)$, and Newton non-degenerate. Suppose that all compact faces τ_0 verify*

(i) $\frac{k(\tau_0)}{N(\tau_0)} < 1$.

(ii) For every face τ of codimension 1 of $\Gamma(f)$, distinct of τ_0 and having non-empty intersection with τ_0, we have $\varepsilon(\tau_0, \tau) \notin \mathbb{Z}$.

Then, the real parts of the poles of the zeta function of f are roots of the Bernstein–Sato polynomial of f.

As explained, this is based on a cohomological argument of existence of a non-zero class, which in turn forces the two stated conditions, the second one basically ensuring that the monodromies are not identity.

Remark 3.17*.* Loeser already points out in [16, Remarque 5.5.2.1] that if one replaces the condition $\frac{k(\tau_0)}{N(\tau_0)} < 1$ with $\frac{k(\tau_0)}{N(\tau_0)} \notin \mathbb{N}$, this is enough to prove the weak version of the conjecture.

As for the second hypothesis, although it is not clear if it is possible to remove, we can try to relax it. Indeed, one ought to expect that non-positive residue numbers could be allowed to happen, after possibly generalizing results in the spirit of [7, Proposition 2.14] or [4, Proposition 11.1]. Therefore, we will next try to construct examples where this is the case, and analyze the contributions of such divisors violating the condition.

3.4. Constructing counterexamples to the second condition

To begin, it should be mentioned that all examples studied have been checked to satisfy the topological strong monodromy conjecture (and thus the weak version too). Nonetheless, the relevant findings are examples where the second condition on the (toric) residue numbers is not met, and even more, where ε is a positive integer.

Even more interestingly, these examples have been motivated geometrically from the Newton polyhedron. More precisely, f is first constructed by introducing monomials $x^p + y^q + z^r$ for p, q, r large enough integers. Then, adding small mixed monomials of the type $x^s y^t z^u$ for small enough integers s, t, u, we obtain small compact faces close to the origin. By choosing the exponents appropriately, we can construct a polyhedron whose faces have normal vectors as desired. In particular, we can find pairs of adjacent faces for which the corresponding divisors of the resolution give rise to (toric) residue numbers that are integers.

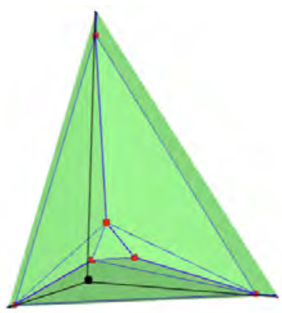

Figure 2: Example of the construction of $\Gamma(x^5 + y^6 + z^7 + x^2yz + xyz^2 + xy^2z)$.

As an illustrative example, we consider $f = x^5 + y^6 + z^7 + x^2yz + xyz^2 + xy^2z$, whose Newton diagram is depicted in the above Figure 2. The original rays in the dual fan are:

$$[(0,0,1),(0,1,0),(1,0,0),(1,1,1),(1,1,2),(1,2,1),(3,1,1),(6,5,14),(7,18,5),(23,7,6)],$$

and a regular subdivision requires almost 400 new rays. Also, the Bernstein–Sato polynomial is

$$b_f(s) = \left(s+\tfrac{11}{6}\right)\left(s+\tfrac{9}{5}\right)\left(s+\tfrac{12}{7}\right)\left(s+\tfrac{5}{3}\right)\left(s+\tfrac{8}{5}\right)\left(s+\tfrac{11}{7}\right)\left(s+\tfrac{3}{2}\right)\left(s+\tfrac{10}{7}\right)\left(s+\tfrac{7}{5}\right)$$
$$\cdot\left(s+\tfrac{4}{3}\right)\left(s+\tfrac{9}{7}\right)\left(s+\tfrac{5}{4}\right)\left(s+\tfrac{6}{5}\right)\left(s+\tfrac{7}{6}\right)\left(s+\tfrac{8}{7}\right)(s+1)^3\left(s+\tfrac{6}{7}\right)\left(s+\tfrac{5}{6}\right)\left(s+\tfrac{4}{5}\right)\left(s+\tfrac{3}{4}\right),$$

and the (local) topological zeta function (which is the same expression as the global version Theorem 3.14, but the second sum runs only over compact faces) is

$$Z_0(s) = \frac{81/4}{s+3/4} - \frac{72/5}{s+4/5} - \frac{70/6}{s+5/6} - \frac{48/7}{s+6/7} + \frac{14}{s+1},$$

with poles $\{-3/4, -4/5, -5/6, -6/7, -1\}$. Therefore, it is immediate that the strong monodromy conjecture holds for this case.

However, we have the ray $(1,1,2)$ with candidate pole value $\sigma = -k/N = -4/5$ appearing, and which is a bad divisor. Indeed, we have the (toric) residue number

$$\varepsilon((1,1,2),(7,8,5)) = 2.$$

In light of this, the next natural question is whether we can discard the contribution of this *bad* divisor to the zeta function, and work only with those satisfying both conditions (in the spirit of the case of plane curves, where a study of the dual graph allowed to discard non-rupture divisors for which we don't have the cohomological result).

Following the above example, the two divisors with candidate pole value $-4/5$ are the bad divisor $(1,1,2)$ and also $(1,2,1)$. So we compute the contributions of each divisor to the (local) topological zeta function, by summing only the terms from cones Δ_i that contain the ray as one of its generating rays. As a warning, for a ray a this is not the same as simply taking the terms where a fraction $\frac{1}{N(a)s+k(a)}$ appears, as it can happen that another ray a' gives rise to the same candidate pole.

In particular, we find that the residues of the individual contributions are

$$\operatorname*{Res}_{s=-4/5} Z_{0;(1,2,1)}(s) = -\frac{47}{5}, \quad \operatorname*{Res}_{s=-4/5} Z_{0;(1,1,2)}(s) = -\frac{47}{5}$$

so we can't discard the bad divisor.

Remark 3.18. Another remark to point here is that the total residue won't necessarily be the sum of these residues. Indeed, there can be cones where both rays appear as generating rays (this happens precisely if the associated divisors intersect), and in that case we would need to subtract the doubly counted contribution. More generally, an inclusion-exclusion formula should be applied in order to compare the separate contributions and the total one for divisors with the same candidate pole value.

Altogether, the reasoning and study of the presented example confirms that we can't omit the additional hypothesis on the residue numbers required for the proof of the conjecture in the Newton non-degenerate case and, even worse, that the possible divisors not satisfying it can indeed contribute with non-zero residue to the poles of the topological zeta function. In other words, the approach by Loeser based on the construction of a non-zero cohomology class via the mentioned results does not allow to extend the proof of the conjecture to the general case.

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