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Convergence of generalized MIT bag models

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Resum *(CAT)*
Estudiem propietats espectrals dels models de bossa de l'MIT generalitzats. Aquests són operadors de Dirac $\{\mathcal{H}_\tau\}_{\tau\in\mathbb{R}}$ actuant en dominis de $\mathbb{R}^3$ amb condicions de frontera que generen confinament. Estudiant la convergència en el sentit de la resolvent dels operadors $\mathcal{H}_\tau$ cap als operadors límit $\mathcal{H}_{\pm\infty}$ quan $\tau \to \pm\infty$, provem que certes propietats espectrals s'hereden al llarg de la parametrització. Aquests resultats, obtinguts parcialment al treball de fi de màster [3], són nous i s'han publicat a [4].

Abstract *(ENG)*
We study spectral properties of generalized MIT bag models. These are Dirac operators $\{\mathcal{H}_\tau\}_{\tau\in\mathbb{R}}$ acting on domains of $\mathbb{R}^3$ with confining boundary conditions. By studying the resolvent convergence of the operators $\mathcal{H}_\tau$ towards the limiting operators $\mathcal{H}_{\pm\infty}$ as $\tau \to \pm\infty$, we prove that certain spectral properties are inherited throughout the parametrization. These results, partially obtained in the master's thesis [3], are new and have been published in [4].

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1. Introduction

The equation that governs all relativistic quantum processes is called *Dirac equation*. In $\mathbb{R}^3$, it is a system of four complex valued linear PDEs of first order in time and space variables. For a spin-1/2 free particle of mass $m \geq 0$, one can write the Dirac equation in matricial form as

$$i\frac{\partial}{\partial t}\psi(x,t) = (-i\alpha\cdot\nabla + m\beta)\psi(x,t), \quad x\in\mathbb{R}^3,\ t\geq 0, \tag{1}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and β are the so-called *Dirac matrices*,

$$\alpha_j := \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} \text{ for } j = 1,2,3, \quad \text{and} \quad \beta := \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \quad \text{with} \quad I_2 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

given by the *Pauli matrices*

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and where

$$\psi(x,t) = \begin{pmatrix} \psi_1(x,t) \\ \psi_2(x,t) \\ \psi_3(x,t) \\ \psi_4(x,t) \end{pmatrix} \in \mathbb{C}^4$$

is the so-called *wave function* of the particle. Here, $\nabla = (\partial_1, \partial_2, \partial_3)$ denotes the gradient in $\mathbb{R}^3$, and as customary we use the notation $\alpha\cdot\nabla = \alpha_1\partial_1 + \alpha_2\partial_2 + \alpha_3\partial_3$. In Cartesian coordinates, the differential operator in the right-hand side of (1) writes as

$$-i\alpha\cdot\nabla + m\beta = \begin{pmatrix} m & 0 & -i\partial_3 & -i\partial_1 - \partial_2 \\ 0 & m & -i\partial_1 + \partial_2 & i\partial_3 \\ -i\partial_3 & -i\partial_1 - \partial_2 & -m & 0 \\ -i\partial_1 + \partial_2 & i\partial_3 & 0 & -m \end{pmatrix}.$$

Notice that if one diagonalizes this operator (taking into account boundary conditions), one can solve the time-dependent Dirac equation (1) using the method of separation of variables. Hence, the time-dependent problem reduces to a stationary eigenvalue problem of the form

$$\begin{cases} (-i\alpha\cdot\nabla + m\beta)\varphi = \lambda\varphi & \text{in } \Omega, \\ \text{boundary conditions for } \varphi & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subseteq \mathbb{R}^3$ is the domain where the particle evolves, $\varphi\colon \Omega \to \mathbb{C}^4$, and the boundary conditions typically depend on physical constraints. The eigenvalues λ provide relevant information to understand the evolution of the system, hence this motivates their study and understanding. This is what we do in this work, for some prescribed boundary conditions.

2. Generalized MIT bag models

Dirac operators acting on domains $\Omega \subset \mathbb{R}^3$ with C^2 boundary are used in relativistic quantum mechanics to describe particles that are confined in a box. The so-called *MIT bag model* is a very remarkable example, which was introduced in the 1970s as a simplified model to study confinement of quarks in hadrons (like quarks up and down inside a proton). It is the operator $\mathcal{H}_0$ defined by

$$\begin{aligned}\mathrm{Dom}(\mathcal{H}_0) &:= \{\varphi \in H^1(\Omega) \otimes \mathbb{C}^4 : \varphi = -i\beta(\alpha \cdot \nu)\varphi \text{ on } \partial\Omega\},\\ \mathcal{H}_0\varphi &:= (-i\alpha \cdot \nabla + m\beta)\varphi \quad \text{for all } \varphi \in \mathrm{Dom}(\mathcal{H}_0).\end{aligned}$$

Here, ν denotes the outward unit normal vector on $\partial\Omega$, and $H^1(\Omega)$ is the standard Sobolev space of first weak derivatives in $L^2(\Omega)$, namely

$$H^1(\Omega) := \{f \in L^2(\Omega) : \|f\|_{H^1(\Omega)} < \infty\}, \quad \text{where} \quad \|f\|_{H^1(\Omega)} := (\|f\|^2_{L^2(\Omega)} + \|\nabla f\|^2_{L^2(\Omega)})^{1/2}.$$

For the sake of notation, in the sequel we shall denote $H^1(\Omega) \otimes \mathbb{C}^4$ as $H^1(\Omega)^4$, and similarly $L^2(\Omega) \otimes \mathbb{C}^4$ as $L^2(\Omega)^4$.

Motivated by some physical considerations, in [1] it was studied the family of Dirac operators with confining boundary conditions defined for $\tau \in \mathbb{R}$ by

$$\begin{aligned}\mathrm{Dom}(\mathcal{H}_\tau) &:= \{\varphi \in H^1(\Omega)^4 : \varphi = i(\sinh\tau - \cosh\tau\,\beta)(\alpha \cdot \nu)\varphi \text{ on } \partial\Omega\},\\ \mathcal{H}_\tau\varphi &:= (-i\alpha \cdot \nabla + m\beta)\varphi \quad \text{for all } \varphi \in \mathrm{Dom}(\mathcal{H}_\tau).\end{aligned} \tag{2}$$

Notice that the MIT bag model corresponds to $\tau = 0$ —this was the main reason in [1] to call the operators $\mathcal{H}_\tau$ in (2) *generalized MIT bag models*. For $\tau \in \mathbb{R}$, the operator $\mathcal{H}_\tau$ is self-adjoint in $L^2(\Omega)^4$ by [2, Proposition 5.15]. Moreover, from [1, Lemma 1.2] we know that its spectrum $\sigma(\mathcal{H}_\tau)$ is contained in $\mathbb{R} \setminus [-m, m]$ and is purely discrete. In particular, the essential spectrum $\sigma_{\mathrm{ess}}(\mathcal{H}_\tau)$ is empty for all $\tau \in \mathbb{R}$. Furthermore, $\lambda \in \sigma(\mathcal{H}_\tau)$ if and only if $-\lambda \in \sigma(\mathcal{H}_{-\tau})$. Thanks to this odd symmetry, one can reduce the study of the spectral properties of the generalized MIT bag models to the study of $\sigma(\mathcal{H}_\tau) \cap (m, +\infty)$ for $\tau \in \mathbb{R}$.

A spectral study of the mapping $\tau \mapsto \mathcal{H}_\tau$ was carried out in [1], where the following result was shown. In its statement, $-\Delta_D$ denotes the self-adjoint realization of the Dirichlet Laplacian in $L^2(\Omega)$, and $\sigma(-\Delta_D)$ denotes its spectrum.

Theorem 2.1 ([1, Theorem 1.4]). *The eigenvalues of $\mathcal{H}_\tau$ can be parametrized by increasing real analytic functions of τ. Moreover, if $\tau \mapsto \lambda(\tau) \in \sigma(\mathcal{H}_\tau) \cap (m, +\infty)$ is a continuous function defined on an interval $I \subset \mathbb{R}$, then the following holds:*

(i) If $I = (-\infty, \tau_0)$ for some $\tau_0 \in \mathbb{R}$, then $\lambda(-\infty) := \lim_{\tau\downarrow-\infty} \lambda(\tau)$ exists and belongs to $[m, +\infty)$. In addition,

$$\lambda(-\infty) = \begin{cases} m \text{ if } \lambda(\tau) \leq \sqrt{\min\sigma(-\Delta_D) + m^2} \text{ for some } \tau \in I,\\ \sqrt{\lambda_D + m^2} \text{ for some } \lambda_D \in \sigma(-\Delta_D) \text{ otherwise.}\end{cases}$$

(ii) If $I = (\tau_0, +\infty)$ for some $\tau_0 \in \mathbb{R}$, then $\lambda(+\infty) := \lim_{\tau\uparrow+\infty} \lambda(\tau)$ exists as an element of the set $(m, +\infty]$. In addition, if $\lambda(+\infty) < +\infty$, then

$$\lambda(+\infty) = \sqrt{\lambda_D + m^2} \quad \text{for some } \lambda_D \in \sigma(-\Delta_D).$$

This result establishes a clear connection between the spectrum of the Dirac operator $\mathcal{H}_\tau$ as $\tau \to \pm\infty$ and the spectrum of the Dirichlet Laplacian $-\Delta_D$. In [1, Remark 4.4] it was left as an open question to investigate which should be the limiting operators of $\mathcal{H}_\tau$ as $\tau \to \pm\infty$, and in which sense the convergence holds true. The answer was developed in the master's thesis [3] and then published in [4]. In the present work, we review the results obtained.

3. Convergence as τ moves in $\mathbb{R}$

In order to guess who the limiting operators might be, we first make an observation. Writing $\varphi \in \mathrm{Dom}(\mathcal{H}_\tau)$ in components[1] as $\varphi = (u, v)^\intercal$, the boundary condition

$$\varphi = i(\sinh\tau - \cosh\tau\,\beta)(\alpha\cdot\nu)\varphi$$

rewrites as $u = -ie^{-\tau}(\sigma\cdot\nu)v$. Formally, this equation forces u and v to vanish on $\partial\Omega$ in the limits $\tau \uparrow +\infty$ and $\tau \downarrow -\infty$, respectively. This leads to consider the so-called *Dirac operators with zigzag type boundary conditions* studied in [6], which are defined by

$$\begin{aligned} \mathrm{Dom}(\mathcal{H}_{+\infty}) &:= \{\varphi = (u, v)^\intercal : u \in H^1_0(\Omega)^2,\ v \in L^2(\Omega)^2,\ \alpha\cdot\nabla\varphi \in L^2(\Omega)^4\},\\ \mathcal{H}_{+\infty}\varphi &:= (-i\alpha\cdot\nabla + m\beta)\varphi \quad \text{for all } \varphi \in \mathrm{Dom}(\mathcal{H}_{+\infty}) \end{aligned} \tag{3}$$

—here $H^1_0(\Omega)^2$ is the subspace of functions in $H^1(\Omega)^2$ with zero trace—, and

$$\begin{aligned} \mathrm{Dom}(\mathcal{H}_{-\infty}) &:= \{\varphi = (u, v)^\intercal : u \in L^2(\Omega)^2,\ v \in H^1_0(\Omega)^2,\ \alpha\cdot\nabla\varphi \in L^2(\Omega)^4\},\\ \mathcal{H}_{-\infty}\varphi &:= (-i\alpha\cdot\nabla + m\beta)\varphi \quad \text{for all } \varphi \in \mathrm{Dom}(\mathcal{H}_{+\infty}). \end{aligned} \tag{4}$$

From [6, Theorem 1.1 and Lemma 3.2] we know that $\mathcal{H}_{\pm\infty}$ are self-adjoint in $L^2(\Omega)^4$ and that their spectra are characterized by the spectrum of the Dirichlet Laplacian. More specifically,

$$\begin{aligned} \sigma(\mathcal{H}_{+\infty}) &= \{-m\} \cup \{\pm\sqrt{\lambda_D + m^2} : \lambda_D \in \sigma(-\Delta_D)\},\\ \sigma(\mathcal{H}_{-\infty}) &= \{m\} \cup \{\pm\sqrt{\lambda_D + m^2} : \lambda_D \in \sigma(-\Delta_D)\}, \end{aligned} \tag{5}$$

and $\mp m \in \sigma_{\mathrm{ess}}(\mathcal{H}_{\pm\infty})$ is an eigenvalue of infinite multiplicity.

Observe that the description (5) of $\sigma(\mathcal{H}_{\pm\infty})$ is in agreement with the limiting spectrum stated in Theorem 2.1. This heuristically motivates to propose the operators $\mathcal{H}_{\pm\infty}$ defined in (3) and (4) as the limiting operators of $\mathcal{H}_\tau$, as $\tau \to \pm\infty$. To see in which sense the convergence holds true, we study the resolvent convergence of $\mathcal{H}_\tau$ to $\mathcal{H}_{\pm\infty}$ as $\tau \to \pm\infty$; see [8, Chapter 8] for a survey on resolvent convergence.

Theorem 3.1 ([4, Theorem 1.2])**.** *Given $\tau \in \mathbb{R}$, let $\mathcal{H}_\tau$ be the operator defined in* (2)*. Let $\mathcal{H}_{+\infty}$ and $\mathcal{H}_{-\infty}$ be the operators defined in* (3) *and* (4)*, respectively. Then, $\mathcal{H}_\tau$ converges to $\mathcal{H}_{\pm\infty}$ in the strong resolvent sense as $\tau \to \pm\infty$. That is, for every $f \in L^2(\Omega)^4$*

$$\lim_{\tau\to\pm\infty} \|((\mathcal{H}_{\pm\infty} - \lambda)^{-1} - (\mathcal{H}_\tau - \lambda)^{-1})f\|_{L^2(\Omega)^4} = 0 \quad \textit{for all } \lambda \in \mathbb{C}\setminus\mathbb{R}. \tag{6}$$

[1] The notation $\varphi = (u, v)^\intercal$ refers to the decomposition of $\varphi\colon \Omega \to \mathbb{C}^4$ in upper and lower components, that is, if $\varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4)^\intercal$ with $\varphi_j\colon \Omega \to \mathbb{C}$ for $j = 1, 2, 3, 4$, then $u = (\varphi_1, \varphi_2)^\intercal$ and $v = (\varphi_3, \varphi_4)^\intercal$.

A proof of this theorem based on directly estimating the difference of resolvents in (6) can be found in [3, Section 3.2], and an alternative proof based on the notion of strong graph limit [8, Definition in p. 293] can be found both in [3, Section 3.1] and in [4, Section 2]. An immediate consequence of this theorem is the following result, which is an improvement of item (ii) in Theorem 2.1 for the first positive eigenvalue of $\mathcal{H}_\tau$.

Corollary 3.2 ([4, Corollary 1.3])**.** *For every $\tau \in \mathbb{R}$, denote the first positive eigenvalue of $\mathcal{H}_\tau$ in Ω by $\lambda_\Omega(\tau) := \min(\sigma(\mathcal{H}_\tau) \cap (m, +\infty))$. Then, $\lim_{\tau\uparrow+\infty} \lambda_\Omega(\tau) = \sqrt{\Lambda_\Omega + m^2}$, where $\Lambda_\Omega := \min \sigma(-\Delta_D)$ is the first eigenvalue of the Dirichlet Laplacian in Ω.*

It is remarkable to point out that Theorem 3.1 does not ensure that the convergence in (6) is uniform in the unit ball of $L^2(\Omega)^4$, but only pointwise for every $f \in L^2(\Omega)^4$. Actually, we now justify that the convergence can not be uniform —in the language of resolvents, this means that $\mathcal{H}_\tau$ can not converge to $\mathcal{H}_{\pm\infty}$ in the norm resolvent sense as $\tau \to \pm\infty$; see [8, Definition in p. 284] or Theorem 3.4 below—: indeed, if there was convergence in the norm resolvent sense, [9, Satz 9.24] would lead to $\lim_{\tau\to\pm\infty} \sigma_{\text{ess}}(\mathcal{H}_\tau) = \sigma_{\text{ess}}(\mathcal{H}_{\pm\infty})$, but this is impossible since $\sigma_{\text{ess}}(\mathcal{H}_{\pm\infty}) \neq \emptyset$ —recall that $\mp m$ is an eigenvalue of infinite multiplicity— and $\sigma_{\text{ess}}(\mathcal{H}_\tau) = \emptyset$ for all $\tau \in \mathbb{R}$ —because $\sigma(\mathcal{H}_\tau)$ is purely discrete.

This argument shows that the essential eigenvalue $\mp m \in \sigma_{\text{ess}}(\mathcal{H}_{\pm\infty})$ prevents $\mathcal{H}_\tau$ from converging to $\mathcal{H}_{\pm\infty}$ in the norm resolvent sense. It is then natural to ask whether the norm resolvent convergence could be achieved if, in some sense, the study was restricted to $\sigma(\mathcal{H}_{\pm\infty}) \setminus \{\mp m\}$. An affirmative answer holds true in the following sense. Denote

$$\ker(\mathcal{H}_{\pm\infty} \pm m) := \{\psi \in \operatorname{Dom}(\mathcal{H}_{\pm\infty}) \subset L^2(\Omega)^4 : (\mathcal{H}_{\pm\infty} \pm m)\psi = 0\},$$
$$\ker(\mathcal{H}_{\pm\infty} \pm m)^\perp := \{\varphi \in L^2(\Omega)^4 : \langle \varphi, \psi \rangle_{L^2(\Omega)^4} = 0 \text{ for all } \psi \in \ker(\mathcal{H}_{\pm\infty} \pm m)\}.$$

Since $\ker(\mathcal{H}_{\pm\infty} \pm m)^\perp$ is a closed subspace of $L^2(\Omega)^4$, the orthogonal projection

$$P_\pm \colon L^2(\Omega)^4 \to \ker(\mathcal{H}_{\pm\infty} \pm m)^\perp \subset L^2(\Omega)^4 \tag{7}$$

is a well-defined bounded self-adjoint operator in $L^2(\Omega)^4$. Moreover, from (5) we know that $\ker(\mathcal{H}_{\pm\infty} \pm m)^\perp \neq \{0\}$ and, thus, $\|P_\pm\|_{L^2(\Omega)^4 \to L^2(\Omega)^4} = 1$.

Theorem 3.3 ([4, Theorem 1.4])**.** *Given $\tau \in \mathbb{R}$, let $\mathcal{H}_\tau$ be the operator defined in (2). Let $\mathcal{H}_{+\infty}$ and $\mathcal{H}_{-\infty}$ be the operators defined in (3) and (4), respectively. Then,*

$$\lim_{\tau\to\pm\infty} \|P_\pm((\mathcal{H}_{\pm\infty} - \lambda)^{-1} - (\mathcal{H}_\tau - \lambda)^{-1})\|_{L^2(\Omega)^4 \to L^2(\Omega)^4} = 0 \quad \text{for all } \lambda \in \mathbb{C} \setminus \mathbb{R},$$

where $P_\pm$ are the orthogonal projections defined in (7).

A proof of this theorem can be found in [4, Section 3]. As we mentioned after Corollary 3.2, the difference of resolvents $(\mathcal{H}_{\pm\infty} - \lambda)^{-1} - (\mathcal{H}_\tau - \lambda)^{-1}$ does not converge to zero in operator norm as $\tau \to \pm\infty$. However, if we write this difference as

$$(\mathcal{H}_{\pm\infty} - \lambda)^{-1} - (\mathcal{H}_\tau - \lambda)^{-1} = (P_\pm + (1 - P_\pm))((\mathcal{H}_{\pm\infty} - \lambda)^{-1} - (\mathcal{H}_\tau - \lambda)^{-1}),$$

then Theorem 3.3 shows that the eigenvalue $\mp m$ is indeed the only obstruction for having norm resolvent convergence of $\mathcal{H}_\tau$ to $\mathcal{H}_{\pm\infty}$ as $\tau \to \pm\infty$, since $(1 - P_\pm)(L^2(\Omega)^4) = \ker(\mathcal{H}_{\pm\infty} \pm m)$.

Although the main interest is the study of the convergence of $\mathcal{H}_\tau$ in a resolvent sense as $\tau \to \pm\infty$, for the sake of completeness we also study the convergence when τ approaches a finite value $\tau_0 \in \mathbb{R}$.

Theorem 3.4. *Given $\tau \in \mathbb{R}$, let $\mathcal{H}_\tau$ be the operator defined in (2). Then, for every $\tau_0 \in \mathbb{R}$, $\mathcal{H}_\tau$ converges to $\mathcal{H}_{\tau_0}$ in the norm resolvent sense as $\tau \to \tau_0$. That is,*

$$\lim_{\tau\to\pm\infty} \|(\mathcal{H}_{\pm\infty} - \lambda)^{-1} - (\mathcal{H}_\tau - \lambda)^{-1}\|_{L^2(\Omega)^4\to L^2(\Omega)^4} = 0 \quad \textit{for all } \lambda \in \mathbb{C}\setminus\mathbb{R}.$$

A proof of this theorem based on the fact that the resolvent operator $(\mathcal{H}_\tau - \lambda)^{-1}$ is real analytic in τ in a neighborhood of τ_0 —given by [1, Lemma 3.1]— can be found in [3, Section 3.4]. An alternative proof based on estimating the operator norm of the difference of resolvents can be found in [4, Section 4].

4. Shape optimization

A hot open problem in spectral geometry is to prove that the first positive eigenvalue $\lambda_\Omega(\tau)$ of $\mathcal{H}_\tau$ is minimal, among all bounded C^2 domains $\Omega \subset \mathbb{R}^3$ with prescribed volume, when Ω is a ball; see [1, Conjecture 1.8]. The analogous statement for the first eigenvalue of the Dirichlet Laplacian, $\Lambda_\Omega := \min \sigma(-\Delta_D)$, is known to be true, and it is the so-called *Faber–Krahn inequality* —proven independently by Faber in 1923 and Krahn in 1925 [5, 7], asserting that $\Lambda_\Omega > \Lambda_B$ whenever $\Omega \subset \mathbb{R}^3$ is a bounded domain with Lipschitz boundary different from a ball B with the same volume.

As an application of the results obtained in [3, 4] and presented in this paper, we conclude with a statement supporting (but not proving) the optimality of the ball for $\lambda_\Omega(\tau)$. On the one hand, $\tau \mapsto \lambda_\Omega(\tau)$ is an increasing and continuous function in $\mathbb{R}$, that converges to m as $\tau \to -\infty$ —by Theorem 2.1— and that converges to $\sqrt{\Lambda_\Omega + m^2}$ as $\tau \uparrow +\infty$, by Corollary 3.2; in particular, $\tau \mapsto \lambda_\Omega(\tau)$ is bijective from $\mathbb{R}$ to $(m, \sqrt{\Lambda_\Omega + m^2})$. On the other hand, if Ω is not a ball, then by the Faber–Krahn inequality we have $(m, \sqrt{\Lambda_B + m^2}) \subsetneq (m, \sqrt{\Lambda_\Omega + m^2})$. Therefore, there exists a large enough $\tau_\Omega \in \mathbb{R}$ such that $\lambda_\Omega(\tau) \in (\sqrt{\Lambda_B + m^2}, \sqrt{\Lambda_\Omega + m^2})$ for all $\tau \geq \tau_\Omega$. Since $\lambda_B(\tau) < \sqrt{\Lambda_B + m^2}$ for all such τ —by Theorem 2.1 and Corollary 3.2—, we get the following.

Proposition 4.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with C^2 boundary, and let B be a ball such that $|\Omega| = |B|$. If Ω is not a ball, then there exists $\tau_\Omega \in \mathbb{R}$ such that $\lambda_B(\tau) < \lambda_\Omega(\tau)$ for all $\tau \geq \tau_\Omega$.*

It is very remarkable to say that the large enough $\tau_\Omega \in \mathbb{R}$ ensuring the optimality of the ball for the first positive eigenvalue $\lambda_\Omega(\tau)$ in the regime $\tau \geq \tau_\Omega$ depends itself on Ω. Hence, from Proposition 4.1 one can *not* ensure that there exists a large enough $\tau_\star \in \mathbb{R}$ for which $\lambda_\Omega(\tau) > \lambda_B(\tau)$ for all $\tau \geq \tau_\star$ and *every* bounded C^2 domain Ω different from a ball B with the same volume. To prove or disprove the existence of such $\tau_\star$ also remains as an open and challenging problem.

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