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On the basins of attraction of root-finding algorithms

David Rosado Rodríguez
Universitat de Barcelona
rosadodav4@gmail.com

Resum *(CAT)*
Els algoritmes de cerca d'arrels s'han utilitzat per resoldre numèricament equacions no lineals de la forma $f(x) = 0$. Aquest article estudia la dinàmica de la família parametritzada de Traub $T_{p,\delta}$ aplicada a polinomis, que abasta des del mètode de Newton ($\delta = 0$) fins al de Traub ($\delta = 1$). Ens centrem en propietats topològiques de les conques immediates d'atracció dels punts fixos finits, especialment la seva connectivitat i el fet de ser acotada o no. Aquests fets són clau per identificar condicions inicials universals que assegurin la convergència a totes les arrels de p.

Abstract *(ENG)*
Root-finding algorithms have historically been used to numerically solve nonlinear equations of the form $f(x)=0$. This paper studies the dynamics of the parameterized Traub family $T_{p,\delta}$ applied to polynomials, ranging from Newton's method ($\delta = 0$) to Traub's method ($\delta = 1$). We focus on topological properties of the immediate basins of attraction of the finite fixed points, especially simple connectivity and unboundedness, which are key to identifying a universal set of initial conditions ensuring convergence to all roots of p.

Keywords: *dynamical systems, root-finding algorithms, Newton's method, Traub's method.*
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1. Introduction

Solving non-linear equations of the form $f(x) = 0$ is a common challenge in various scientific fields, spanning from biology to engineering. When algebraic manipulation is not feasible, iterative methods become necessary to determine solutions. Among these, Newton's method stands out as a widely used approach, relying on the linearization of $f(x)$. Its iterative scheme is given by:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n \geq 0.$$

Nevertheless, numerous numerical methods have proven to be efficient when they converge, including the one considered here, Traub's method. While Newton's method exhibits quadratic convergence for simple roots of a polynomial when the initial guess is sufficiently close to the root, Traub's method achieves cubic (local) convergence. This method belongs to a parametric family of iterative schemes, first introduced in [6, 12], known as the *damped Traub's family*. Its iterative formula is given by:

$$x_{n+1} = y_n - \delta \frac{f(y_n)}{f'(x_n)}, \quad n \geq 0,$$

where $y_n = x_n - \frac{f(x_n)}{f'(x_n)}$ represents a Newton step, and δ is the damping parameter. Notably, setting $\delta = 0$ recovers Newton's method, while $\delta = 1$ corresponds to Traub's method. It is important to mention that each iteration of Traub's method involves additional computations compared to Newton's method. Although the question can be explored in other settings, here we will focus on the case where $p(z) = 0$, $z \in \mathbb{C}$.

Roughly speaking, when we have a good estimate of the solutions to the equation $p(z) = 0$, iterative methods tend to work well. However, challenges arise when the number of solutions of f is large or when we lack control over these solutions. This is particularly problematic when selecting initial conditions to initiate the algorithm. In such situations, the study of dynamical systems becomes valuable. By examining the topological characteristics of the immediate basins of attraction associated with the solutions of $p(z) = 0$, we can gain valuable insights and aid in addressing these challenges. An illustration of this is provided in [8]. In their work, the authors used some topological results of the basins of attraction to construct a universal and explicit set of initial conditions denoted as $\mathcal{S}_d$. This set, depending only on the polynomial's degree, allows Newton's method to find all roots of a polynomial. The existence of the set $\mathcal{S}_d$ is guaranteed by the following key properties of the immediate basins of attractions for the Newton's method (first proved by Przytycki [10] and later generalized by Shishikura [11]).

Theorem 1.1. *Let p be a polynomial of degree $d \geq 2$. Assume that $p(\alpha) = 0$ and let N_p be the corresponding Newton's map. Then, the immediate basin of attraction of α, denoted as $\mathcal{A}^*(\alpha)$, is a simply connected, unbounded set.*

A natural question that comes up now is whether we can create a set similar to $\mathcal{S}_d$ for Traub's method. If this were possible, it would provide a way to find all the roots of a polynomial with improved convergence speed. Specifically, as previously noted, for simple roots of the polynomial, the local convergence order would be cubic instead of quadratic, leading to faster convergence. To achieve this, proving an equivalent to Theorem 1.1 for Traub's method, will provide the necessary tools for building the $\mathcal{S}_d$ like-set. In a recent study [3], Theorem 1.1 was proved for Traub's method under certain additional assumptions. To be precise, the researchers successfully established the following theorem:

Theorem 1.2. *Let p be a polynomial of degree $d \geq 2$. Assume that p satisfies one of the following conditions:*

(i) $d = 2$, or

(ii) it can be written in the form $p_{n,\beta}(z) := z^n - \beta$ for some $n \geq 3$ and $\beta \in \mathbb{C}$.

Suppose that $p(\alpha) = 0$ and consider damped Traub's map $T_{p,\delta}(z) := N_p(z) - \delta \frac{p(N_p(z))}{p'(z)}$ with $\delta \in [0,1]$. Then $\mathcal{A}^*_\delta(\alpha)$ is a simply connected unbounded set.

This article explores the damped Traub's method as a root-finding algorithm. We provide background to understand the proof of Theorem 1.2 and present two results that bring us closer to our goal: proving an equivalent result to Theorem 1.1 for Traub's method. Specifically, we establish the following result:

Theorem A. *Let p be a polynomial of degree $d \geq 2$. Assume that $p(\alpha) = 0$ and let $T_{p,\delta}$ be the corresponding damped Traub's map. Then, for δ close enough to zero, $\mathcal{A}^*_{T_{p,\delta}}(\alpha)$ is an unbounded set.*

Moreover, we analyze the behavior of Traub's method for the polynomial family $p_d(z) = z(z^d - 1)$. Notably, for Halley's root-finding algorithm, the basin of attraction of $z = 0$ is bounded when $d = 5$, but we establish the following result:

Theorem B. *Let $p_d(z) = z(z^d - 1)$. Then, $\mathcal{A}^*_{T_{p_d,1}}(0)$ is an unbounded set for every integer $d > 0$.*

2. An introduction to holomorphic dynamics

Let us denote $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ the *extended complex plane* or *Riemann Sphere*.

Definition 2.1. Let $R\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map. A point $z = z_0$ is a *fixed point* if $R(z_0) = z_0$ (resp. *periodic of period p* if $R^p(z_0) = z_0$ for some $p \geq 1$ and $R^n(z_0) \neq z_0$ for all $n < p$). The *multiplier* of z_0 is $\lambda = R'(z_0)$ (resp. $\lambda = (R^p)'(z_0)$).

The character of the fixed or periodic points can be determined by using the multiplier. In fact, the fixed or periodic point z_0 is *attracting* if $|\lambda| < 1$ (*superattracting* if $\lambda = 0$), *repelling* if $|\lambda| > 1$ and *indifferent* if $|\lambda| = 1$.

Definition 2.2. Let $R\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map and $z_0 \in \hat{\mathbb{C}}$ be an attracting fixed point of R. We define the *basin of attraction* of z_0 as

$$\mathcal{A}_R(z_0) := \mathcal{A}(z_0) = \{z \in \hat{\mathbb{C}} \colon R^n(z) \xrightarrow{n\to\infty} z_0\}.$$

We denote by $\mathcal{A}^*(z_0)$ the connected component of $\mathcal{A}(z_0)$ containing z_0, and we refer to it as the *immediate basin of attraction*.

In what follows we omit the dependence with respect to the rational map under consideration, unless it is mandatory. It is easy to see that $\mathcal{A}(z_0)$ is an open set containing z_0. There is a vast body of results on this topic, and for a general overview, many excellent references are available; see, for instance, [2, 9, 5]. To conclude this chapter, we present a theorem that will be useful later. This theorem state that, in a neighbourhood of an attracting fixed point, the map *looks like* $g(\zeta) = \lambda\zeta$.

Theorem 2.3 (Koenigs linearization Theorem). *Let $z_0 \in \mathbb{C}$, U neighbourhood of z_0 and $f: U \to \mathbb{C}$ be a holomorphic function such that z_0 is an attracting fixed point with multiplier $0 < |\lambda| < 1$. Then there is a conformal map $\zeta = \phi(z)$ of a neighbourhood of z_0 onto a neighbourhood of z_0 which conjugates f to the linear function $g(\zeta) = \lambda\zeta$. The conjugating function is unique, up to multiplication by a nonzero scale factor.*

3. Local dynamics of the family $T_{p,\delta}$

Recall that if p is a polynomial of degree $d \geq 2$, the damped Traub's map applied to p is defined as

$$T_{p,\delta} = N_p(z) - \delta \frac{p(N_p(z))}{p'(z)},$$

where N_p is the Newton's map and $\delta \in \mathbb{C}$. For our purposes, it will suffice to consider $\delta \in [0, 1]$. Notice that, setting $\delta = 0$ recovers the well-known Newton's method. The map N_p is the *universal* root-finding algorithm and it satisfies the following key global dynamical properties:

Proposition 3.1. *Let p be a polynomial of degree $d \geq 2$. The following properties regarding the Newton's map hold:*

(i) A point $z = \alpha$ is a root of p if and only if it is a fixed point of N_p.

(ii) The simple roots of p are superattracting fixed points of N_p, while multiple roots are attracting fixed points of N_p.

(iii) The point $z = \infty$ is the only repelling fixed point of N_p.

Proof. (i) and (ii) are straightforward computations. For (iii), observe that $N_p(\infty) = \lim_{z\to\infty} N_p(z) = \infty$, so $z = \infty$ is a fixed point. To see its nature, consider the transformation $\phi: U \to V$ such that $\phi(z) = 1/z$, where U is a neighbourhood of $z = \infty$ and V is a neighbourhood of $z = 0$. The conjugate map is then $\widetilde{N_p}(z) = \phi(N_p(\phi^{-1}(z))) = \frac{1}{N_p(1/z)}$, so studying the behaviour of $\widetilde{N_p}$ at $z = 0$ reveals the character of $z = \infty$ in the original system. □

Leveraging the properties of Newton's method, particularly noting that $z = \infty$ is the only repelling fixed point, Theorem 1.1 can be established. Details of the proof can be found in [1]. For $\delta \neq 0$, some of the properties that hold for N_p remain the same, while others change slightly. Let us summarize them:

Proposition 3.2. *Let p be a polynomial of degree $d \geq 2$ and $\delta \in (0, 1]$. The following properties regarding the damped Traub's map hold:*

(i) If $z = \alpha$ is a root of p, then $z = \alpha$ is a fixed point of $T_{p,\delta}$. The converse is not necessarily true.

(ii) The simple roots of p are superattracting fixed points of $T_{p,\delta}$, while multiple roots are attracting fixed points of $T_{p,\delta}$.

(iii) The point $z = \infty$ is a repelling fixed point of $T_{p,\delta}$.

The complete and detailed proof can be found in [3]. Notice that, finite fixed points of the method do not necessarily correspond to zeros of the polynomial, as is the case with Newton's method; see Figure 1 for a visual illustration. With all this information, a recent study [3] successfully established Theorem 1.2.

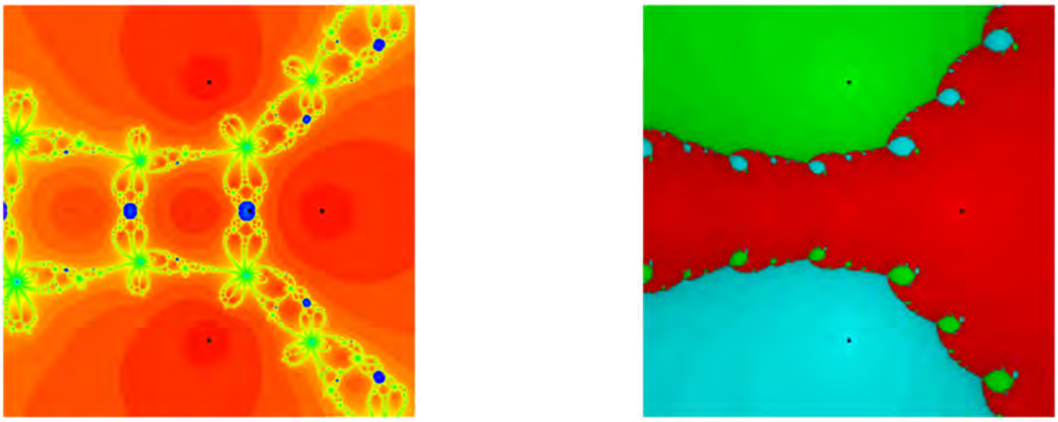

Figure 1: On the left, we illustrate the dynamical plane of Traub's method applied to the cubic polynomial $P(z) = (z^2 + 0.25)(z - 0.439)$. Basins of attraction corresponding to roots of the polynomial are shown in orange. It is notable that $T_{p,1}$ exhibits an attracting fixed point located at $\zeta \approx 0.155$, whose basin is depicted in blue, that does not correspond to any root of P. On the right, we present the dynamical plane of Newton's method applied to P. Here, it is evident that there are no fixed points other than the roots.

4. The method as a singular perturbation

In this chapter, we prove the unboundedness of immediate basins of attraction when $\delta \approx 0$. For small δ, the damped Traub's method acts as a singular perturbation of Newton's method. A *singular perturbation* refers to a base family (with well-understood dynamics) combined with a local perturbation that increases the map's degree and enriches its dynamics. This perturbation affects only certain regions of the dynamical plane when the parameter is small [7]. Here, Newton's method is the base family, and $p(N_p(z))/p'(z)$ is the perturbation. Notice that the singular perturbation occurs over the Julia set, as it involves adding additional preimages of $z = \infty$ to the zeros of $p'(z)$. To establish the main result of the section, we will first present some auxiliary results.

Lemma 4.1. *Let $p(z) = a_d z^d + \cdots + a_1 z + a_0$ be a polynomial of degree $d \geq 2$. Let q_j be the zeros of $p'(z) = 0$, i.e., the poles of both the damped Traub's map, $T_{p,\delta}$, and Newton's map, N_p. Consider the compact $K = \overline{D(0, R)} \setminus \cup_j D(q_j, \varepsilon)$ where $R > 0$ and $\varepsilon > 0$ are positive fixed constants. Then, for every $z \in K$, there exists a constant $C_{R,\varepsilon}$ such that $|p(N_p(z))/p'(z)| \leq C_{R,\varepsilon}$.*

Proof. Let $z \in K$. There exists a positive value $\eta_\epsilon > 0$ such that $|p'(z)| > \eta_\epsilon$. Moreover, since $|z| < R$, $|p(z)| \leq |a_d|R^d + \cdots + |a_1|R + |a_0| := R'$. Hence,

$$|N_p(z)| \leq |z| + \left|\frac{p(z)}{p'(z)}\right| \leq R + \frac{R'}{\eta_\epsilon} := M.$$

Therefore,

$$\left|\frac{p(N_p(z))}{p'(z)}\right| \leq \frac{|a_d N_p(z)^d| + \cdots + |a_1 N_p(z)| + |a_0|}{\eta_\epsilon} \leq \frac{|a_d|M^d + \cdots + |a_1|M + |a_0|}{\eta_\epsilon} := C_{R,\varepsilon}. \qquad \square$$

Lemma 4.2. *Let p be a polynomial of degree $d \geq 2$. Let q_j be the zeros of $p'(z) = 0$, i.e., the poles of both the damped Traub's map, $T_{p,\delta}$, and Newton's map, N_p, and let $z = \alpha$ be a zero of p, i.e., an attracting fixed point for both N_p and $T_{p,\delta}$. Consider the compact $K = \overline{D(0,R)} \setminus \cup_j D(q_j, \varepsilon')$ where $R > 0$ and $\varepsilon' > 0$ are positive fixed constants such that $\alpha \in K$. Then, the following statements hold:*

(i) There exists a compact $K' \subset K$ such that $K' \subset \mathcal{A}^*_{N_p}(\alpha)$, $\alpha \in K'$ and $\partial K' \cap \partial K \neq \emptyset$, satisfying that for every $z \in K'$, there is a unique $M \in \mathbb{N}$ such that: $\forall \varepsilon > 0$, $N_p^M(z) \in D(\alpha, \varepsilon/2)$.

(ii) For the given $\varepsilon > 0$ and for δ small enough, the following property holds: $\forall z \in K'$, $|N_p^M(z) - T_{p,\delta}^M(z)| < \varepsilon/2$. In particular, $T_{p,\delta}^M(z) \in D(\alpha, \varepsilon)$.

Proof. (i) The existence of such a compact is guaranteed by the fact that $\mathcal{A}^*_{N_p}(\alpha)$ is an open set, unbounded and simply connected. Since $z = \alpha$ is an attracting fixed point for N_p, the existence of $M \in \mathbb{N}$ is also guaranteed.

(ii) To prove the result, let us first establish the following claim: For δ small enough,

$$\forall r > 0,\ \exists \rho > 0 \text{ such that if } |z_1 - z_2| < \rho \implies |N_p(z_1) - T_{p,\delta}(z_2)| < r.$$

To prove the claim, observe that, using Lemma 4.1 in the last inequality,

$$|N_p(z_1) - T_{p,\delta}(z_2)| \leq |N_p(z_1) - N_p(z_2)| + \delta \left| \frac{p(N_p(z))}{p'(z)} \right| \leq |N_p(z_1) - N_p(z_2)| + \delta C_{R,\varepsilon'}.$$

Hence, since Newton's map is continuous in K (in particular it is also in K'), there exists $\rho > 0$ such that if $|z_1 - z_2| < \rho$, then $|N_p(z_1) - N_p(z_2)| < r/2$. Setting $\delta = \frac{r}{2C_{R,\varepsilon'}}$, we obtain the desired bound.

Now, let $z \in K$. To prove the result, we proceed as follows:

1. Using the claim with $z_1 := N_p^{M-1}(z)$ and $z_2 := T_{p,\delta}^{M-1}(z)$, there exists $\eta_M > 0$ and $\delta_M > 0$ such that if $|N_p^{M-1}(z) - T_{p,\delta}^{M-1}(z)| < \eta_M$, then $|N_p^M(z) - T_{p,\delta_M}^M(z)| < \varepsilon/2$.
2. Iterating the algorithm, we obtain sequences $\{\eta_{M-i}\}_{i=0}^{M-3}$, $\{\delta_{M-i}\}_{i=0}^{M-3}$ satisfying that if $|N_p^{M-i-1}(z) - T_{p,\delta}^{M-i-1}(z)| < \eta_{M-i}$, then $|N_p^{M-i}(z) - T_{p,\delta_i}^{M-i}(z)| < \eta_{M-i+1}$.
3. We conclude the algorithm with the existence of $\eta_2 > 0$ and $\delta_2 > 0$ such that if $|N_p(z) - T_{p,\delta_2}(z)| < \eta_2$, then $|N_p^2(z) - T_{p,\delta_2}^2(z)| < \eta_3$.

Finally, to ensure that $|N_p(z) - T_{p,\delta}(z)| < \eta_2$, we just need to take $\delta_1 = \frac{\eta_2}{C_{R,\varepsilon'}}$. Therefore, taking $\delta = \min\{\delta_1, \dots, \delta_M\}$, we obtain that for every $z \in K$: $|T_{p,\delta}^M(z) - N_p^M(z)| < \varepsilon/2$. ☐

Theorem A. *Let p be a polynomial of degree $d \geq 2$. Assume that $p(\alpha) = 0$ and let $T_{p,\delta}$ be the corresponding damped Traub's map. Then, for δ close enough to zero, $\mathcal{A}^*_{T_{p,\delta}}(\alpha)$ is an unbounded set.*

Proof. First, observe that for δ close enough to zero (indeed for every $\delta \in [0,1]$), $z = \infty$ is a repelling fixed point for $T_{p,\delta}$. By Koenigs linearization Theorem, in a neighborhood of $z = \infty$, say $D(\infty, \varepsilon)$, $T_{p,\delta}$ is locally conjugated to $g(\zeta) = \lambda\zeta$, where λ is the multiplier of $z = \infty$. Notice that, if $\lambda \in \mathbb{C}$, since $|\lambda| > 1$, points near $z = \infty$ tend to move away in a spiral shape, and if $\lambda \in \mathbb{R}$, since $|\lambda| > 1$, points near $z = \infty$ tend to move outward in a radial manner.

Let us define $R := \frac{1}{\varepsilon}$ and consider the compact $K := \overline{D(0, R)} \setminus \cup_j D(q_j, \varepsilon')$ where q_j are the poles of $T_{p,\delta}$, i.e., the zeros of $p'(z) = 0$, and $\varepsilon' > 0$ is a positive fixed constant. We can assume that $\alpha \in K$. If not, we can choose a smaller value for ε (increasing the value of R) to ensure that $\alpha \in K$, making the neighborhood where the Koenigs coordinates apply smaller. Since $z = \alpha$ is an attracting fixed point for both N_p and $T_{p,\delta}$, there exists $\eta_1, \eta_2 > 0$ such that $D(\alpha, \eta_1) \subset \mathcal{A}^*_{N_p}(\alpha)$ and $D(\alpha, \eta_2) \subset \mathcal{A}^*_{T_{p,\delta}}(\alpha)$. Setting $\eta = \min\{\eta_1, \eta_2\}$, we have that $D(\alpha, \eta) \subset \mathcal{A}^*_{N_p}(\alpha) \cap \mathcal{A}^*_{T_{p,\delta}}(\alpha)$. According to Lemma 4.2(i), there exists a compact $K' \subset K$ such that $K' \subset \mathcal{A}^*_{N_p}(\alpha)$, $\alpha \in K'$ and $\partial K' \cap \partial K \neq \emptyset$, satisfying that for every $z \in K'$, there is a unique $M \in \mathbb{N}$ such that, for every $z \in K'$, $N_p^M(z) \in D(\alpha, \eta/2) \subset D(\alpha, \eta)$. Moreover, since the basins of attraction of Newton's method are unbounded and simply connected, there exists a ray τ connecting the fixed point $z = \alpha$ and $z = \infty$, included in $\mathcal{A}^*_{N_p}(\alpha)$. This ray can be chosen such that its restriction to K is included in K'. From now own, any reference to τ will indicate the ray extending from the point $z = \alpha$ to the boundary of the set K. Then, according to Lemma 4.2(ii), for δ small enough and $z \in K'$, $T^M_{p,\delta}(z) \in D(\alpha, \eta)$, indicating that $z \in \mathcal{A}^*_{T_{p,\delta}}(\alpha)$. Then, either $\tau \subset K' \subset \mathcal{A}^*_{T_{p,\delta}}(\alpha)$ or there exists $w \in \mathcal{J}(T_{p,\delta}) \cap K'$. In the last case, since $w \in K'$, $T^M_{p,\delta}(w) \in D(\alpha, \eta)$, in contradiction with $w \in \mathcal{J}(T_{p,\delta})$. Therefore, $\tau \subset K' \subset \mathcal{A}^*_{T_{p,\delta}}(\alpha)$.

By construction, observe that $\partial D(0, R) = \partial D(\infty, \varepsilon)$, hence, the ray τ, which ends at $\partial D(0, R)$, connects with the spiral (or the line in case $\lambda \in \mathbb{R}$) that extends towards $z = \infty$. Thus, we found a ray that connects the fixed point $z = \alpha$ to $z = \infty$, which is contained within $\mathcal{A}^*_{T_{p,\delta}}(\alpha)$. This proves that the immediate basin of attraction for the damped Traub's method is unbounded when $\delta \approx 0$. □

5. Traub's method applied to $z(z^d - 1)$

Now, we aim to examine Traub's method applied to the family $p_d(z) = z(z^d - 1)$. This family is particularly interesting because, for Halley's root-finding algorithm, it was found that for $d = 5$, the immediate basin of attraction of $z = 0$ is bounded. Therefore, proving that this is not the case for Traub's method would support the conjecture that the immediate basins of attraction of Traub's method are unbounded. We have been able to prove that the immediate basin of attraction of $z = 0$ is unbounded for every d. To establish the result, we will first present an auxiliary result.

Lemma 5.1. *The semi-lines $z = re^{\frac{(2k+1)\pi i}{d}}$, $r > 0$ and $k = 0, 1, \ldots, d-1$, are forward invariant under $T_{p_d,1}$.*

First of all, observe that

$$T_{p_d,1}(z) = N_{p_d}(z) - \frac{p_d(N_{p_d}(z))}{p'_d(z)} = \frac{d(d+1)z^{2d+1}[(d+1)z^d - 1]^d - [dz^{d+1}]^{d+1}}{[(d+1)z^d - 1]^{d+2}}.$$

Hence, since $e^{(2k+1)\pi i} = -1$, a straightforward computation reveals that $T_{p_d,1}(re^{\frac{(2k+1)\pi i}{d}}) = e^{\frac{(2k+1)\pi i}{d}} R_d(r)$, where

$$R_d(r) := \frac{d(d+1)r^{2d+1}[(d+1)r^d + 1]^d - d^{d+1}r^{(d+1)^2}}{[(d+1)r^d + 1]^{d+2}}.$$

Theorem B. *Let $p_d(z) = z(z^d - 1)$. Then, $\mathcal{A}^*_{T_{p_d,1}}(0)$ is an unbounded set for every integer $d > 0$.*

Proof. Consider only the semi-lines that do not cross the dth roots of unity, i.e., $z = re^{\frac{(2k+1)\pi i}{d}}$, $r > 0$ and $k = 0, 1, \ldots, d-1$. By Lemma 5.1, the semi-lines are forward invariant under $T_{p_d,1}$. In fact, we have that $T_{p_d,1}(re^{\frac{(2k+1)\pi i}{d}}) = e^{\frac{(2k+1)\pi i}{d}} R_d(r)$, where

$$R_d(r) := \frac{d(d+1)r^{2d+1}[(d+1)r^d + 1]^d - d^{d+1}r^{(d+1)^2}}{[(d+1)r^d + 1]^{d+2}}.$$

Then, if we can prove that for every $r > 0$ we have $0 < R_d(r) < r$, we can conclude that $\mathcal{A}^*_{T_{p_d,1}}(0)$ is an unbounded set for every d. In that case, we can also state that $\mathcal{A}^*_{T_{p_d,1}}(0)$ has at least d accesses to infinity. Since the denominator of R_d is always positive for every $r > 0$, the inequality $0 < R_d(r)$ is equivalent to

$$d(d+1)r^{2d+1}[(d+1)r^d + 1]^d - d^{d+1}r^{(d+1)^2} > 0. \tag{1}$$

Expanding the last expression using the Binomial expansion, we obtain that inequality (1) becomes

$$d(d+1)\sum_{j=0}^{d-1}\binom{d}{d-j}(d+1)^j r^{dj+2d+1} + d[(d+1)^{d+1} - d^d]r^{(d+1)^2} > 0.$$

Notice that, since $(d+1)^{d+1} - d^d > 0$ for every positive integer d, we obtain that the inequality holds for every $r > 0$.

The inequality $R_d(r) < r$ can be written as $S_d(r) < 0$, where S_d is defined as

$$S_d(r) := d(d+1)r^{2d+1}[(d+1)r^d + 1]^d - d^{d+1}r^{(d+1)^2} - r[(d+1)r^d + 1]^{d+2}.$$

Using the Binomial expansion, we can rewrite the last expression:

$$S_d(r) = d(d+1)\sum_{j=0}^{d-1}\binom{d}{d-j}(d+1)^j r^{dj+2d+1} + d[(d+1)^{d+1} - d^d]r^{(d+1)^2}$$

$$-\sum_{j=-2}^{d}\binom{d+2}{d-j}(d+1)^{j+2}r^{dj+2d+1}.$$

Now, arranging terms,

$$S_d(r) = [d((d+1)^{d+1} - d^d) - (d+1)^{d+2}]r^{(d+1)^2} - \sum_{j=-2}^{0}\binom{d+2}{d-j}(d+1)^{j+2}r^{dj+2d+1}$$

$$+\sum_{j=0}^{d-1}\left[d(d+1)\binom{d}{d-j}(d+1)^j - \binom{d+2}{d-j}(d+1)^{j+2}\right]r^{dj+2d+1}.$$

Observe that $d((d+1)^{d+1} - d^d) - (d+1)^{d+2} = -(d+1)^{d+1} - d^{d+1} < 0$ and

$$d(d+1)\binom{d}{d-j}(d+1)^j - \binom{d+2}{d-j}(d+1)^{j+2} = (d+1)^{j+1}\left[\frac{d!d - (d+2)!(d+1)}{(d-j)!j!}\right] < 0.$$

Hence, all the coefficients of the polynomial S_d are negative. Therefore, we can conclude that for $r > 0$, $S_d(r) < r$, which completes the proof. □

It still needs to be proven that the immediate basins of attraction of the dth roots of unity are unbounded. This is a more challenging part of the proof, as attempting to apply the same arguments used earlier leads to difficulties in establishing bounds for the expressions. However, a recent study confirms that this holds for all integers $d \geq 2$. In fact, the case of Traub's method applied to the family $p(z) = z(z^d - 1)$ has already been fully resolved [4].

6. Conclusions

With this paper, we are contributing towards demonstrating that the immediate basins of attraction of the damped Traub's method are unbounded and simply connected sets. We have been able to prove with complete generality the unboundedness of the method when $\delta \approx 0$ and we analyze a particular case, the family $p_d(z) = z(z^d - 1)$. Our findings indicate that analyzing the topological properties of this method is not a straightforward and that a comprehensive proof requires different approaches from those used in [3].

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