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On the basins of attraction of root-finding algorithms

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Resum *(CAT)*
Els algoritmes de cerca d'arrels han estat històricament utilitzats per resoldre numèricament equacions no lineals de la forma $f(x) = 0$. Aquest treball explora la dinàmica dels mètodes de la família Traub parametritzada $T_{p,\delta}$ aplicada a polinomis. Aquests mètodes inclouen un ventall des del mètode de Newton ($\delta = 0$) fins al mètode de Traub ($\delta = 1$). El nostre enfocament rau a investigar diverses propietats topològiques de les conques d'atracció, particularment la seva simple connectivitat i la no acotació, que són crucials per identificar un conjunt universal de condicions inicials que assegurin la convergència a totes les arrels de p.

Keywords: *dynamical systems, root-finding algorithms, Newton's method, Traub's method.*

Abstract

Solving nonlinear equations of the form $f(x) = 0$ is a common challenge in various scientific fields, spanning from biology to engineering. When algebraic manipulation is not feasible, iterative methods become necessary to determine solutions. Newton's method is a well-known approach, derived from linearizing the equation $f(x) = 0$. Its iterative expression is given by:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n \geq 0.$$

Over the past few decades, numerous researchers have suggested various iterative approaches aimed at enhancing Newton's method [4]. One prevalent strategy for devising new methods involves directly combining existing techniques and subsequently modifying them to minimize the count of functional evaluations. For example, if we apply Newton's method twice while keeping the derivative constant in the second step, we derive Traub's method [5]. A specific type of root-finding algorithms, called the *damped Traub's family*, was first introduced in the papers [2, 6]. Its iterative expression is given by:

$$x_{n+1} = y_n - \delta\frac{f(y_n)}{f'(x_n)}, \quad n \geq 0,$$

where $y_n = x_n - \frac{f(x_n)}{f'(x_n)}$ is a Newton's step and δ is the damping parameter. Notice that $\delta = 0$ corresponds to Newton's method and $\delta = 1$ to Traub's method. Newton's method converges quadratically for simple roots of a polynomial when the initial guess is sufficiently close to the desired root. On the other hand, Traub's method exhibits cubic (local) convergence. It is worth noting that each iteration of Traub's method requires more computations compared to Newton's method.

The challenge of iterative methods lies in the choice of initial conditions to start the algorithm. The study of dynamical systems plays a crucial role in gaining insight into how to make this selection effectively. An example of this is presented in [3], where a universal and explicit set of initial conditions, denoted by $\mathcal{S}_d$, is constructed. This set depends solely on the degree of the polynomial and can be used to find all the roots of a polynomial using Newton's method. The existence of this set is ensured by the fact that the immediate basins of attraction for the method are simply connected and unbounded sets.

The aim of this work is to construct a set analogous to $\mathcal{S}_d$ for Traub's method. If successful, this would offer a way to find all the roots of a polynomial with enhanced convergence speed. To achieve this, it is necessary to prove that the immediate basins of attraction for Traub's method are simply connected and unbounded sets. This would provide the essential framework for constructing a set similar to $\mathcal{S}_d$. A recent study [1] demonstrated this result under the assumptions that the polynomial is either of degree 2, or it can be expressed in the form $p_{n,\beta}(z) := z^n - \beta$, where $n \geq 3$ and $\beta \in \mathbb{C}$.

We contributed to this research by analyzing the behavior of the damped Traub's family when the damping factor is close enough to zero by considering the method as a singular perturbation. We have been successful in proving the unbounded nature of the immediate basins of attractions for this case. Furthermore, we focus on investigating the simple connectivity and unboundedness of the immediate basins of attractions specifically for third-degree polynomials, achieving some findings concerning the distribution of both the free critical points and the fixed points that are not roots for the damped Traub's method under the condition that δ is close to zero. Finally, we conclude our research by examining Traub's method applied to the family $p_d(z) = z(z^d - 1)$. We have proven the unboundedness of the immediate basins of attraction for specific values of d, and we present evidences suggesting that this unboundedness holds for all values of d.

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