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Moduli of plane branches with a single characteristic exponent

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Resum *(CAT)*
Estudiem l'espai de mòduls de branques planes (per equivalència analítica) amb un únic exponent característic mitjançant una estratificació utilitzant el semigrup de valors de l'ideal jacobià de la branca. En particular, estudiem com abordar el problema mitjançant diferents tècniques. En primer lloc, proporcionem un procediment algorítmic basat en un procediment de Casas-Alvero, que, sota alguns supòsits, descriu els estrats. Incloem una implementació en `Maple` d'aquest algorisme. A més, comparem la nostra estratificació amb una altra estudiada prèviament per Peraire l'any 1998 a partir de l'invariant de Zariski. Això ens permet fer algunes reflexions sobre els reptes de calcular la dimensió dels nostres estrats, que refinen els estrats de Peraire, i presentar algunes noves eines per abordar el problema.

Keywords: *analytic classification, stratification, Jacobian ideal.*

Abstract

The moduli problem of classifying by analytical equivalence germs of irreducible curves in the same equisingularity class was first introduced by Oscar Zariski, [3], who gave a partial description of the space and worked out some examples. The difficulty of the problem soon became apparent and many open questions, that remain still open, arose. Our goal is to study the moduli problem for plane branches with a single characteristic exponent and describe a stratification using the semigroup of values of the Jacobian ideal of the branch, denoted by Θ. This stratification refines a previously known one based on the Zariski invariant studied by Rosa Peraire in 1998, [2].

The germs

$$\gamma_{\mathbf{A},\sigma} : f_{\mathbf{A},\sigma} = y^n - x^m + x^p y^q + \sum_{\substack{ni+mj>nm+\sigma \\ 0\le i\le m-2 \\ 0\le j\le n-2}} A_{i,j} x^i y^j = 0, \quad A_{i,j} \in \mathbb{C},$$

where we denote by σ the integer $np + mq - nm$ and by $\mathbf{A}$ the coefficients $A_{i,j}$, represent all analytic types of germs with a single characteristic exponent equal to m/n, $(n, m) = 1$, and Zariski invariant equal to σ. Casas-Alvero in [1] describes a procedure that obtains for a given $f(\mathbf{A}, x, y) \in \mathbb{C}(\mathbf{A})\{x, y\}$, as in the previous equation with fixed Zariski invariant σ, the semigroup of values of the Jacobian ideal of the branch $\gamma_{\mathbf{A}}$ according to a set of conditions on the $A_{i,j}$. These conditions describe the strata that correspond to the Zariski invariant σ of the stratification of the moduli space using the semimodule of values of the

Jacobian ideal of a branch. One of our main contributions is developing this procedure into an algorithm and implementing it in `Maple`. The implementation of this procedure also lead us to the construction of an interesting tree, the tree of conditions for the Jacobian values of a fixed single characteristic exponent m/n. A rooted tree structure in which the leaves represent all the possible semigroups of values of the Jacobian ideal of $\gamma_{\mathbf{A}}$ and the nodes represent the conditions that the set of coefficients must satisfy in each of the cases.

From our algorithm we are able to deduce a semi-reduced and a reduced equation describing of all the branches in any fixed stratum with fixed set of Jacobian values Θ, and as a by-product the dimension of that stratum. Given any stratification, a general equation $f_{\mathbf{A},\sigma}$ of a stratum (representing all its analytic types) is semi-reduced if the only non-null coefficients $A_{i,j}$ are those whose variation results in a change in the analytical type of the branch $\gamma_{\mathbf{A},\sigma}$. A reduced equation of a stratum is a semi-reduced equation expressed in terms of a minimal number of $A_{i,j}$. These $A_{i,j}$ in a reduced equation provide a parametrization of the stratum, and its cardinal is precisely the dimension of that stratum.

Peraire in her Theorem 4.12 of [2] gives a combinatorial expression for the dimension of the strata of her stratification by the Zariski invariant. We generalize her result and prove that an analogous expression accounts for the number of non-null coefficients $A_{i,j}$ in a semi-reduced equation. Furthermore, we prove that for her strata, any semi-reduced equation is, in fact, reduced, which does not hold in general for our strata. We reveal that this presents the main difficulty in providing a combinatorial expression for the dimension of the strata in our stratification by the set of Jacobian values Θ. Finally we introduce the notion of Θ-continuous coefficients, which we believe is strongly related to this dimension.

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