Extended Abstracts

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Spectral gap of generalized MIT bag models

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Resum *(CAT)*
Estudiem propietats espectrals dels models de bossa de l'MIT generalitzats. Aquests són operadors de Dirac $\{\mathcal{H}_\tau\}_{\tau\in\mathbb{R}}$ actuant en dominis de $\mathbb{R}^3$ amb condicions de frontera que generen confinament. El seu autovalor positiu més baix és d'especial interès, i s'ha conjecturat que és mínim per a una bola entre tots els dominis amb volum fixat. La conjectura anàloga és certa per al laplacià de Dirichlet (és la desigualtat de Faber–Krahn), que sorgeix en els límits $\tau \to \pm\infty$. Estudiant la convergència en el sentit de la resolvent dels operadors $\mathcal{H}_\tau$ cap als operadors límit $\mathcal{H}_{\pm\infty}$ quan $\tau \to \pm\infty$, provem que certes propietats espectrals s'hereden al llarg de la parametrització. Aquests resultats són nous i s'han publicat a [3].

Keywords: *Dirac operator, spectral theory, MIT bag model, shape optimization, resolvent convergence.*

Abstract

The equation that governs all relativistic quantum processes is called Dirac equation. In $\mathbb{R}^3$, it is a complex valued system of four linear PDEs of first order in time and space variables. For a spin-1/2 free particle of mass m, one can write the Dirac equation in matricial form as $i\frac{\partial}{\partial t}\psi(x,t) = (-i\alpha\cdot\nabla + m\beta)\psi(x,t)$, where α and β are the so-called Dirac matrices, given —essentially— by the more known Pauli matrices. The stationary eigenvalue problem associated to the Dirac equation is of the form

$$\begin{cases} (-i\alpha\cdot\nabla + m\beta)\,\varphi = \lambda\varphi & \text{in } \Omega,\\ \text{Boundary conditions} & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subseteq \mathbb{R}^3$ is the domain where the particle evolves, $\varphi\colon \Omega \to \mathbb{C}^4$, and the boundary conditions typically depend on physical constraints. The eigenvalues λ of such Dirac operators provide relevant information to understand the evolution of the system, and hence their study is of special interest.

Dirac operators acting on domains $\Omega \subset \mathbb{R}^3$ are used in relativistic quantum mechanics to describe particles that are confined in a box. The so-called *MIT bag model* is a very remarkable example, which was introduced in the 1970s as a simplified model to study confinement of quarks in hadrons.

In [1] it is introduced a family $\{\mathcal{H}_\tau\}_{\tau\in\mathbb{R}}$ of Dirac operators with confining boundary conditions parameterized by $\tau \in \mathbb{R}$; the particular case $\tau = 0$ corresponds to the MIT bag model. Because of this reason, the operators $\mathcal{H}_\tau$ are called *generalized MIT bag models*.

In this work [2], we study some spectral properties of generalized MIT bag models. Their lowest positive eigenvalue is of special interest, and it is conjectured to be minimal for a ball among all domains of the same volume. The analogous conjecture holds true for the Dirichlet Laplacian (it is the Faber–Krahn inequality).

We prove that the Dirichlet Laplacian arises in the limit $\tau \to \pm\infty$ by studying the resolvent convergence of $\mathcal{H}_\tau$ in this limit. More specifically, we show strong resolvent convergence of $\mathcal{H}_\tau$ to $\mathcal{H}_{\pm\infty}$, and we justify that one cannot improve this to norm resolvent convergence. These results are new and have been published in [3], together with other extended results.

Because of this convergence, we show that some spectral properties of the limiting operators $\mathcal{H}_{\pm\infty}$ are inherited throughout the parameterization. As a consequence, we verify the conjecture for large enough values of the parameter τ.

Finally, we prove that the conjecture holds true for corona domains of relatively small hole. This result is also new. However, a continuation of this study after the master's thesis —using more abstract arguments— allowed to complete this result for any corona of the same volume (independently of the size of its hole). This extended result will be sent for publication in an indexed journal.

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References

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