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A promenade through singular symplectic geometry

A la memòria del Marc Herault

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Resum *(CAT)*
En aquest article, presentem la geometria simplèctica i de Poisson des de la mecànica hamiltoniana. Després introduïm els algebroides de Lie simplèctics, objectes al mig de la geometria simplèctica i de Poisson. Posteriorment, recordem la noció de reducció simplèctica en presència d'una aplicació moment. Com a aplicació d'aquesta construcció, descrivim els espais de fase de partícules carregades sota la presència de camps de Yang–Mills. Finalment, introduïm un anàleg singular d'aquesta construcció i donem exemples físics.

Abstract *(ENG)*
In this article, we present symplectic and Poisson geometry from the perspective of Hamiltonian mechanics. We then introduce symplectic Lie algebroids, objects which lie between symplectic and Poisson manifolds. Afterwards, we recall the notion of symplectic reduction under the existence of a moment map. As an application of this construction, we describe the phase space of a charged particle interacting with a Yang–Mills field. Finally, we introduce a singular analogue of this construction and provide physical examples.

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1. Introduction

Classical mechanics was inaugurated by the works of Isaac Newton. After his contribution, different approaches to write Newton's equations of motion were developed, commonly with the goal of improving certain aspects of the previous formalism. In Hamiltonian mechanics, the equations of motion are a system of first order ordinary differential equations, known as Hamilton's equations. This feature makes easier discussing qualitative aspects of solutions from the perspective of dynamical systems. Moreover, the dual behavior of symmetries and conserved quantities, originally established by Emmy Noether for Lagrangian systems, becomes transparent in the Hamiltonian formalism.

Symplectic geometry can be regarded as an abstraction of Hamiltonian dynamics for smooth manifolds. Poisson geometry is a further generalization of the symplectic setting, where the relevant structure is the Poisson bracket defining the evolution of observables along the dynamics of the system. As we will see, Poisson structures vastly generalize symplectic structures and, consequently, many results from the symplectic category fail to be transferred to Poisson manifolds. Symplectic Lie algebroids define Poisson structures which, although not arising from a symplectic form, have a very close behavior to them. In physics, these objects allow to describe physical systems with degenerate or constrained dynamics. In mathematics, they have proved to be the adequate language to establish results for a class of Poisson structures.

New discoveries in particle physics during the XX century posed the problem of incorporating the weak and strong forces into mechanics. The satisfactory formulation was proposed by Yang and Mills, and is nowadays known under the name of *gauge theory*.[1] The equations describing the motion of a charged particle under the presence of a Yang–Mills field are a generalization of Lorentz's force equation, and are known as Wong's equations. Sternberg showed how Wong's equations fit into the Hamiltonian formalism of mechanics. Weinstein additionally proved that the phase space constructed by Sternberg could be realized as the reduction of a universal space for particles interacting with Yang–Mills fields.

The goal of this article is to fill the picture introduced in this section. In Section 2 we recall the fundamentals of symplectic and Poisson geometry from the Hamiltonian formalism of mechanics. In Section 3 we introduce Lie algebroids and E-symplectic manifolds as objects between symplectic and Poisson structures. We will additionally give examples of interest where they have been fruitfully applied. In Section 4 we remember the interplay between conserved quantities and symmetries, codified in the moment map of a Hamiltonian action. The presence of symmetries allows for elimination of degrees of freedom, a procedure formalized by the reduction theorem of Marsden and Weinstein. We present Sternberg's and Weinstein's constructions, and show how they have been extended to the setting of E-symplectic manifolds.

2. Symplectic and Poisson geometry

2.1 Symplectic geometry

Symplectic geometry can be considered an abstraction of the Hamiltonian formulation of classical mechanics. In this formalism, the equations of motion in the Euclidean space $\mathbb{R}^{2n}$, described in terms of

[1] In mathematics, gauge theories refer to the study of connections in vector and principal bundles. The name of *Yang–Mills theories* is reserved to the study of solutions to the Yang–Mills equations.

coordinates p_i, q_i, can be recovered from a function $H \in \mathcal{C}^\infty(\mathbb{R}^{2n})$, called the Hamiltonian, following Hamilton's equations of motion:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}. \tag{1}$$

The previous set of equations can be compactly written using matrix notation as $X_H = \mathbb{J} \cdot \nabla f$, where $\mathbb{J}$ is the standard skew-symmetric matrix. Common choices in physics for the Hamiltonian are energy functions of the form $H = \frac{1}{2m}\sum p_i^2 + V(q)$ for some smooth function $V \in \mathcal{C}^\infty(\mathbb{R}^{2n})$, called the potential of the system.

In many examples, as in systems with constraints, it is better to work directly in the setting of differentiable manifolds. To write the previous set of equations in an abstract manifold, however, we need to choose additional data relating the Hamiltonian vector field X_H and the differential $\mathrm{d}H$. Equation (1) suggests that we should choose a skew-symmetric and non-degenerate tensor $\omega \in \Omega^2(M)$. For many results to hold we have to additionally impose the form ω to be closed. While there is good geometric motivation behind this requirement, we do not have the space to delve into this matter.

Definition 2.1. Let M be a smooth manifold. A non-degenerate, closed two-form $\omega \in \Omega^2(M)$ is called a *symplectic form*. We call any such pair (M, ω) a *symplectic manifold*.

Following the previous analogy between ω and the matrix $\mathbb{J}$, Hamilton's equations of motion (1) should be written in this new language as

$$\iota_{X_H}\omega = -\mathrm{d}H.$$

There is no apparent reason to believe this expression should be related in general to equations (1). It is a theorem of Darboux that this is, indeed, the case. More precisely, we have the following:

Theorem 2.2 (Darboux)**.** *Let (M, ω) be a symplectic manifold. For every point $p \in M$ there exists a chart $\varphi\colon U \subset \mathbb{R}^n \to M$ centered at p with coordinates q_i, p_j such that*

$$\varphi^*\omega = \sum_{i=1}^n \mathrm{d}q_i \wedge \mathrm{d}p_i.$$

This result is very powerful because it shows that symplectic geometry has no local invariants. Consequently, all interesting information in symplectic manifolds has to be of topological/global nature.

2.2 Poisson geometry

Poisson brackets were originally introduced to study the evolution of observables, i.e., smooth functions, along the Hamiltonian dynamics. In more mathematical terms, if we define the Poisson bracket of H and f to be the derivative of f along the flow of X_H, Hamilton's equations (1) directly show

$$\{H, f\} = \sum_{i=1}^n \frac{\partial H}{\partial p_i}\frac{\partial f}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial f}{\partial p_i}. \tag{2}$$

In the more general setting of symplectic geometry there exists an analogue generalization of the Poisson bracket given by the formula

$$\{f, g\} = \omega(X_f, X_g). \tag{3}$$

Poisson showed that his eponymous bracket (2) is linear in both arguments, skew-symmetric, satisfies Leibniz's rule and Jacobi's identity holds:

$$\{f,\{g,h\}\}+\{g,\{h,f\}\}+\{h,\{f,g\}\}=0.$$

Even though any bracket arising from (3) satisfies these conditions, there are brackets fulfilling these properties which cannot be defined in this way. A trivial example is the Poisson bracket $\{f,g\}=0$ for all $f,g\in\mathcal{C}^\infty(M)$. The systematic study of these objects is the branch of *Poisson geometry*.

Definition 2.3. A *Poisson bracket* on a smooth manifold M is a bilinear, skew-symmetric operation $\{\cdot,\cdot\}\colon \mathcal{C}^\infty(M)\times\mathcal{C}^\infty(M)\to\mathcal{C}^\infty(M)$ satisfying Leibniz's rule in each argument and Jacobi's identity.

There is an alternative and useful characterization of Poisson brackets. Given any bracket $\{\cdot,\cdot\}\colon \mathcal{C}^\infty(M)\times \mathcal{C}^\infty(M)\to\mathcal{C}^\infty(M)$ satisfying Leibniz's rule and linearity in each variable, we can recover its action on any functions $f,g\in\mathcal{C}^\infty(M)$ as the contraction of a two-tensor field $\Pi\in\mathcal{T}^2M$ with the differentials $\mathrm{d}f$, $\mathrm{d}g$. Moreover, because the Poisson bracket is skew-symmetric, there exists a unique bivector field $\Pi\in\mathfrak{X}^2(M)$ representing the bracket $\{\cdot,\cdot\}$ in the sense that

$$\{f,g\}=\langle \mathrm{d}f\wedge\mathrm{d}g,\Pi\rangle$$

for any smooth functions $f,g\in\mathcal{C}^\infty(M)$. Jacobi's identity, however, does not hold for general bivector fields. It turns out to be equivalent the integrability condition $\mathcal{J}=[\Pi,\Pi]=0$. The trivector field $\mathcal{J}$, up to a factor, is appropriately called the *Jacobiator*, and the bracket $[\cdot,\cdot]$ is an extension of the Lie bracket of vector fields to the space of all multivector fields called *Schouten–Nijenhuis bracket*.

Given the great generality of these structures, there is no local normal form for Poisson structures similar to Darboux's Theorem 2.2. The closest analogue is the following result due to Weinstein.

Theorem 2.4 (Weinstein [15])**.** *Let (M,Π) be a Poisson manifold. For every point $p\in M$ there exists a chart $\varphi\colon U\subseteq M\to\mathbb{R}^n$ with coordinates q_i, p_j, r_k such that*

$$\varphi_*\Pi=\sum_{i=1}^{k}\frac{\partial}{\partial q_i}\wedge\frac{\partial}{\partial p_i}+\sum_{i,j=1}^{n-2k}f_{ij}(r_l)\frac{\partial}{\partial r_i}\wedge\frac{\partial}{\partial r_j}.$$

Moreover, the functions f_{ij} are skew-symmetric and vanish at 0.

This local structure theorem is commonly called the *splitting theorem* because it states that, locally, every Poisson manifold splits as the direct product of a symplectic manifold and a Poisson manifold with vanishing Poisson structure at the origin. Observe this transverse Poisson structure measures the difference of a Poisson manifold from being symplectic.

We would like to highlight two immediate consequences from Weinstein's theorem. Firstly, the splitting shows that any Poisson manifold admits a foliation by symplectic leaves, called the *symplectic foliation* of the manifold. This shows that part of the Poisson structure can be encoded in the symplectic structures of the leaves. Secondly, there is a well-defined notion of *transverse* Poisson structure. In contrast with the symplectic realm, Poisson manifolds do have local invariants. As such, their study is much more complicated.

3. Singular symplectic geometry

We have seen that the class of symplectic manifolds fits very naturally within the class of Poisson manifolds. The class of Poisson manifolds is, however, much bigger and wilder than that of symplectic manifolds. As such, there are some interesting and nice results in the symplectic category that do not hold in Poisson geometry. One instance of this phenomenon is hinted in the difference between Darboux's Theorem 2.2 and Weinstein's Theorem 2.4.

There are many specific examples of Poisson manifolds which, although not being symplectic, can be understood in a symplectic flavour if we are willing to work with singularities. Take as an example the simplest degenerate Poisson structure with its dual form,

$$\Pi = z\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial t} + \sum_{i=2}^{n} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i},$$

$$\omega = \frac{\mathrm{d}z}{z} \wedge \mathrm{d}t + \sum_{i=2}^{n} \mathrm{d}x_i \wedge \mathrm{d}y_i.$$

The form ω is clearly not a symplectic form, because it is not even well-defined as a smooth differential form. It becomes a symplectic form, in some sense, if we restrict its domain to the space of vector fields tangent to the hypersurface $\{z = 0\}$.

We can informally call ω a singular symplectic form. The objective of this section is to elevate this idea to a rigorous statement. We begin by defining the main objects of the discussion.

Definition 3.1. A *Lie algebroid* is a vector bundle $\pi\colon \mathcal{A} \to M$ together with a vector bundle map $\rho\colon \mathcal{A} \to \mathrm{T}M$ covering the identity and equipped with a Lie bracket $[\cdot,\cdot]_{\mathcal{A}}$ on the space of sections $\Gamma\mathcal{A}$. Moreover, the bracket satisfies, for any $X, Y \in \Gamma A$ and $f \in \mathcal{C}^\infty(M)$, the following compatibility conditions:

$$[X, fY]_{\mathcal{A}} = f[X, Y]_{\mathcal{A}} + \mathcal{L}_{\rho(X)} f \cdot Y, \tag{5a}$$

$$\rho([X, Y]_{\mathcal{A}}) = [\rho(X), \rho(Y)]. \tag{5b}$$

In equation (5a), the operator $\mathcal{L}$ denotes the Lie derivative of a function along a vector field.

Equation (5a) is a generalized Leibniz's identity for the bracket. Equation (5b) turns out to be redundant, as it can be deduced from (5a). We have chosen to explicitly state it because it will be relevant for upcoming discussions.

Let us take a brief detour and precisely describe how these objects arise in the description of systems with singularities. We will present examples arising from physics where all the following assumptions are satisfied. Consider that the equations of motion of our system can be described in terms of a $\mathcal{C}^\infty$ subsheaf of vector fields $\mathcal{F} \subseteq \mathfrak{X}$. Furthermore, assume the sheaf is locally finitely generated, that is, for any point $p \in M$ there is an open set U containing p and sections $X_1, \dots, X_m \in \mathcal{F}_U$ such that their restriction to any open set $V \subseteq U$ generates $\mathcal{F}_V$. We can make two additional assumptions, each of which gives rise to well-known objects in differential geometry.

- If we additionally assume the integrability condition $[\mathcal{F}, \mathcal{F}] = \mathcal{F}$, the sheaf $\mathcal{F}$ defines a *singular foliation* in the sense of Androulidakis and Skandalis. These objects can be integrated to give standard singular foliations, or foliations in the sense of Stefan and Sussman.

- If the sheaf is not only locally finitely generated but also locally free, it is a theorem of Serre [11] in the algebraic setting and Swan [13] in the continuous case shows the sheaf $\mathcal{F}$ can be recovered as the sheaf of sections of a vector bundle E.

If both assumptions are simultaneously made, we get *projective foliations* or *Debord foliations*. If E is a representing vector bundle for $\mathcal{F}$ in the sense that $\mathcal{F} = \Gamma E$, we get a natural map of vector bundles $\rho\colon E \to \mathrm{T}M$ given by the evaluation of a section at a point. This map is called the anchor, and is injective in an open and dense subset $U \subseteq M$, i.e., generically injective. The integrability condition $[\mathcal{F}, \mathcal{F}] \subseteq \mathcal{F}$ lifts to a bracket in the space of sections ΓE. One can easily check that the compatibility conditions (5a) and (5b) are satisfied and, thus, any such object is an instance of a Lie algebroid.

Not all Lie algebroids arise this way as, in general, the anchor map $\rho\colon \mathcal{A} \to \mathrm{T}M$ is not generically injective. The class of algebroids previously presented will be relevant in upcoming sections, so we will give them a proper name.

Definition 3.2. Let M be a smooth manifold. An *E-structure* is the choice of a Debord foliation $\mathcal{F} \subseteq \mathfrak{X}$ or, equivalently, a vector bundle $\pi\colon E \to M$ with a generically injective map $\rho\colon E \to \mathrm{T}M$. We call the pair (E, M) an *E-manifold*.

This construction shows that we can consider Lie algebroids, at least psychologically, as a replacement of the standard tangent bundle $\mathrm{T}M$. As such, we can consider the dual bundle $\mathcal{A}^*$ and its exterior powers $\bigwedge^k \mathcal{A}^*$. Sections of these bundles are called, by analogy with the standard setting, *k-differential $\mathcal{A}$-forms*. The space of all sections is written $\Omega^k_{\mathcal{A}}(M)$. The Lie bracket $[\cdot,\cdot]_{\mathcal{A}}$ induces an exterior differential in the spaces $\Omega^k_{\mathcal{A}}(M)$ following the standard Koszul formula,

$$\begin{aligned} \mathrm{d}_{\mathcal{A}}\omega(X_0, \dots, X_k) = &\sum_{i=0}^{k} (-1)^i \mathcal{L}_{\rho(X_i)}\omega(X_0, \dots, \hat{X}_i, \dots, X_k) \\ &+ \sum_{0 \leqslant i < j \leqslant k} (-1)^{i+j}\omega([X_i, X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_k). \end{aligned}$$

In the previous formula, an argument with a hat implies it has been suppressed from the collection of inputs. A routine verification shows $\mathrm{d}^2_{\mathcal{A}} = 0$. The cohomology spaces of the complex of $\mathcal{A}$-forms are called *Lie algebroid cohomology groups*.

With the notion of differential forms and exterior calculus for systems with constraints, we can define a symplectic form mimicking the standard definition in classical differential geometry.

Definition 3.3. Let $\pi\colon \mathcal{A} \to M$ be a Lie algebroid. A *symplectic form* on $\mathcal{A}$ is a two form $\omega \in \Omega^2_{\mathcal{A}}(M)$ which is closed and non-degenerate. We call the pair $(\mathcal{A}, \omega)$ a *symplectic Lie algebroid*. Similarly, if $\pi\colon E \to M$ is an E-manifold, we call the pair (E, ω) an *E-symplectic manifold*.

In this setting, the non-degeneracy condition amounts to requiring the vector bundle morphism

$$\begin{aligned} \omega^\flat\colon \mathcal{A} &\longrightarrow \mathcal{A}^* \\ X &\longmapsto \iota_X\omega \end{aligned}$$

to be an isomorphism. Its inverse map is written $\omega^\sharp\colon \mathcal{A}^* \to \mathcal{A}$. As a consequence, we can define the Hamiltonian vector field associated to a function H as the unique solution to the equation

$$\iota_{X_H}\omega = -\rho^\vee \mathrm{d}H.$$

In the previous equation, the map $\rho^\vee\colon \mathrm{T}^*M \to \mathcal{A}^*$ denotes the adjoint of the anchor map $\rho\colon \mathcal{A} \to \mathrm{T}M$.

We motivated the construction of Lie algebroids and symplectic forms on them to study Poisson structures with certain types of singularities. Any symplectic Lie algebroid indeed defines a Poisson bracket as

$$\{f, g\} = \omega(X_f, X_g).$$

This mapping is clearly bilinear, skew-symmetric, and satisfies Leibniz's identity. Jacobi's identity is not as evident but it is a consequence of the closedness of ω as a singular symplectic form.

Before concluding this section, let us briefly discuss two different examples of E-manifolds which have found success in Poisson geometry.

We begin by describing *b-symplectic manifolds*. In this case, the sheaf of vector fields considered is taken to be the sheaf of tangent vectors to an embedded hypersurface $Z \subseteq M$. The Lie algebroid obtained is called the *b-tangent bundle*. It was originally considered by Melrose [5] in order to generalize the index theorem to manifolds with boundary. The symplectic geometry of b-manifolds has been extensively studied and described by Guillemin, Miranda, and Pires [2]. The school of Miranda has done remarkable work in studying the interplay of b-symplectic geometry with integrable systems, geometric quantization, KAM theory, and many more.

These objects are generalized by b^m-symplectic manifolds. The sheaf of vector fields into consideration is once again the sheaf of all fields tangent to a fixed hypersurface Z, but now we fix with degree of tangency to be at least m. The definition of these structures is due to Scott [10], where some technical details concerning additional data are discussed. This singular symplectic models have found applications in studying the topology of escape orbits in the planar, restricted, circular three body problem [7].

4. Reduction by symmetries and minimal coupling

One of the central ideas in the study of physical systems is that of symmetries. In the presence of a group of transformations that leaves the motion of the system unchanged, one can reduce the number of parameters by an appropriate choice of coordinates (or frames of reference). A remarkable instance of this phenomenon is Euler's solution to the two-body problem. The invariance by linear translations allows the origin of the frame of reference to be taken in the center of mass, while the invariance by rotations implies the confinement of both bodies to a plane and one additional constraint.

These invariance by transformation groups of the system can be dually read as conservation laws. The invariance by linear transformations is equivalent to the conservation of linear momentum, while the invariance by rotations is equivalent to the conservation of angular momentum. The observation that this phenomenon is a general feature is due to Emmy Noether and, as such, the conserved quantities obtained from a symmetry are called *Noether charges*.

This correspondence is transparent in the symplectic formulation of classical mechanics. To describe it, we will need to define what does it mean for an action of a Lie group G on (M, ω) to be Hamiltonian. Intuitively, we would like the fundamental vector fields of the action to be Hamiltonian: in other words, we are asking for a lift $\mu^\bullet$ of the fundamental vector field map $\bullet^\#$ in the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{R} \arrow[r] & \mathcal{C}^\infty(M) \arrow[r, "X_\bullet"] & \mathfrak{X}_{\mathrm{Ham}}(M) \arrow[r] & 0 \\
& & & \mathfrak{g} \arrow[u, "\bullet^\#"'] \arrow[ul, dashed, "\mu^\bullet"] &
\end{tikzcd} \tag{6}$$

Here, we have tacitly assumed M is connected so that $\mathrm{H}^0(M) = \mathbb{R}$. There always exists such a lift at the level of vector spaces: as $X_\bullet\colon \mathcal{C}^\infty(M) \to \mathfrak{X}_{\mathrm{Ham}}(M)$ is surjective, we can choose the preimage of a set of generators and extend by linearity.

There are obstructions for the map $\mu^\bullet\colon \mathfrak{g} \to \mathcal{C}^\infty(M)$ to be a morphism of Lie algebras, where we endow $\mathcal{C}^\infty(M)$ with the Poisson bracket as Lie bracket. The failure to have a Lie algebra morphism is measured by the map

$$c(X, Y) = \{\mu^X, \mu^Y\} - \mu^{[X,Y]}.$$

As the projection of this element to $\mathfrak{X}_{\mathrm{Ham}}(M)$ vanishes by the commutativity of the diagram, we can identify the image $c(X, Y)$ with an element in the kernel $\ker X_\bullet \simeq \mathbb{R}$. This map thus determines an element in the Chevalley–Eilenberg complex, $c \in \mathrm{C}(\mathfrak{g}; \mathbb{R})$. The map c is closed, and hence determines a class in the cohomology group $\mathrm{H}^2(\mathfrak{g}; \mathbb{R})$. The lift $\mu^\bullet$ can be chosen to be a Lie algebra morphism if and only if $[c] = 0$. Moreover, all possible such choices are parametrized by elements of the group $\mathrm{H}^1(\mathfrak{g}; \mathbb{R})$.

Assuming some conditions on the Lie group G,[2] there is a uniquely determined lift which we call the *comoment map*. By construction, it intertwines the adjoint action in $\mathfrak{g}$ with the induced pullback action in $\mathcal{C}^\infty(M)$. As the name hints, however, it is better to think of the comoment map in terms of a dual object called the *moment map*.

Definition 4.1. Let (M, ω) be a symplectic manifold and let G be a Lie group acting on M. We say the action is *Hamiltonian* if there exists a map $\mu\colon M \to \mathfrak{g}^*$, called the *moment map*, satisfying the following conditions:

$$\iota_{X^\#}\omega_p = -\mathrm{d}\langle \mu(p), X\rangle \quad \text{for all } X \in \mathfrak{g},\ p \in M, \tag{7a}$$

$$\mu \circ \rho_g = \mathrm{Ad}_g^* \circ \mu \quad \text{for all } g \in G. \tag{7b}$$

We call any such triple (M, ω, μ) a *Hamiltonian G-space*.

Equation (7a) is reminiscent of the equivariance of the comoment map. Equation (7b) is a direct consequence of the fact that the fundamental vector fields of $\mathfrak{g}$ act in a Hamiltonian fashion: indeed, all we are saying is that $\langle \mu, X\rangle$ is the Hamiltonian function of $X^\#$ or, following (6), the comoment map μ^X.

Let us assume now we are given a Hamiltonian G-space (M, ω, μ) with a G-invariant Hamiltonian H and equations of motion determined by X_H. Assume H is an invariant function under the action of G. We will also assume the group G is connected. This technical condition ensures that any Hamiltonian action is also a symplectic action, that is, preserves the form ω. Two consequences arise from these facts.

Firstly, observe X_H is G-invariant. We recall the symplectic form ω is G-invariant because we assume G is connected. Because the Hamiltonian H is G-invariant, we have

$$\iota_{\rho_g \cdot X_H}\omega = \rho_{g^{-1}}^* \iota_{X_H} \rho_g^* \omega = \rho_{g^{-1}}^* \mathrm{d}H = \mathrm{d}H = \iota_{X_H}\omega.$$

As a consequence, we deduce $X_H = \rho_g \cdot X_H$.

Secondly, assume we are given a regular value $\alpha \in \mathfrak{g}^*$ of μ. By equivariance, every other value $\beta \in \mathcal{O}_\alpha$ is regular, and thus the preimage $M_\alpha = \mu^{-1}(\mathcal{O}_\alpha)$ is a submanifold. By invariance of H we have $\mathcal{L}_{X^\#} H = 0$. The definition of moment map implies now

$$0 = \mathcal{L}_{X_H}\mu^X = \langle \mathrm{d}\mu^X, X_H\rangle = \langle \iota_{X_H}\mu, X\rangle.$$

[2]For example, if G is semisimple, we know by Whitehead's lemmas that $\mathrm{H}^1(\mathfrak{g}; \mathbb{R}) = \mathrm{H}^2(\mathfrak{g}; \mathbb{R}) = 0$. The semisimplicity assumption is automatically satisfied if G is a compact group.

As this equality holds for all $X \in \mathfrak{g}$, we have $\langle \mathrm{d}\mu, X_H \rangle = 0$. This result implies X_H is tangent to the level sets of μ and, by G-invariance, to the preimages of the coadjoint orbits.

Thus, the dynamics defined by X_H can be restricted to the submanifold $\mu^{-1}(\mathcal{O}_\alpha)$ and can be further projected to the quotient manifold $\mu^{-1}(\mathcal{O}_\alpha)/G_\alpha$, assuming technical conditions on the action of G.[3] Marsden and Weinstein observed that this reduced space is once again a symplectic manifold, and hence one can consider Hamiltonian dynamics with respect to the reduced symplectic structure.

Theorem 4.2 (Marsden–Weinstein [4]). *Let (M, ω) be a symplectic manifold and assume G is a compact Lie group acting on M with moment map $\mu\colon M \to \mathfrak{g}^*$. If $\alpha \in \mathfrak{g}^*$ is a regular value of μ, then the space $\mu^{-1}(\mathcal{O}_\alpha)/G_\alpha$ is a symplectic manifold with symplectic form ω_{red}. Moreover, it is uniquely determined by*

$$i_\alpha^*\omega = \pi^*\omega_{\mathrm{red}}.$$

4.1 The minimal coupling procedure

We will describe a procedure to construct the phase space of a charged particle interacting with a Yang–Mills field. The presentation we take is essentially due to Sternberg [12]. Assume we are given a principal G-bundle $\pi\colon P \to X$ over a symplectic manifold (X, ω) and a Hamiltonian G-space (Q, Ω) with moment map μ. We can construct a symplectic structure in the adjoint bundle $P \times_G Q$ by choosing a principal connection $\eta \in \Omega^1(X; \mathrm{ad}\, P)$ in the following way. Sternberg shows the two-form $\mathrm{d}\langle \mu, \eta \rangle + \Omega$ in $P \times Q$ descends to a well-defined and closed two-form $\Omega_\eta \in \Omega^2(P \times_G Q)$. Under a non-degeneracy assumption, the manifold $(P \times_G Q, \omega + \Omega_\eta)$ is symplectic and the previous construction is called the *minimal coupling procedure*. The additional term Ω_η is known in the literature as the *magnetic term*. If we take (Q, Ω) to be a coadjoint orbit of an irreducible representation of a Lie group G, we obtain the classical phase spaces of charged particles [12].

Sternberg mentions that, in the case where $X = \mathrm{T}M$ with its canonical symplectic form, the previous non-degeneracy assumption is always satisfied. Weinstein went beyond this observation and proved Sternberg's phase space can be obtained as the symplectic reduction of a universal phase space. The role of the connection is made explicit in terms of an isomorphism between his construction and Sternberg's. More concretely, we can summarize Weinstein's results in the following theorem.

Theorem 4.3 (Weinstein [14]). *Let $\pi\colon P \to M$ be a principal G-bundle and let (Q, Ω) be a Hamiltonian G-space with moment map μ_Q. Let $P^\#$ be the pullback bundle of π by the submersion $\mathrm{T}^*M \to M$.*

Then, the space $\mathrm{T}^*P \times Q$ is a G-Hamiltonian space for the diagonal G-action with moment map[4] *$\mu = \mu_P + \mu_Q$. Any choice of connection in P induces a diffeomorphism $\mu^{-1}(0) \simeq P^\# \times Q$ which, furthermore, induces a diffeomorphism of the symplectic spaces $\mu^{-1}(0)/G$ and $P^\# \times_G Q$.*

The symplectomorphism induced by this choice of connection is called the *minimal coupling* of the system. Given a Hamiltonian in the base space, $H \in \mathcal{C}^\infty(\mathrm{T}^*M)$, we can consider its pullback to either space and get equivalent dynamics. The induced equations of motion are called *Wong's equations* [8].

[3] Namely, freeness and properness. The latter is satisfied if G is a compact group, which we have enforced since the construction of the moment map.

[4] In this formula, the moment map $\mu_P\colon \mathrm{T}^*P \to \mathfrak{g}^*$ is the natural moment map obtained for the cotangent lift of any G-action to the cotangent bundle with the canonical symplectic form.

There is an interesting interpretation of this construction by Montgomery [9]. The choice of a connection yields the commutative diagram

$$\begin{tikzcd} P^{\#} \times_G Q \arrow[r, "\Phi"] \arrow[dr, "\pi"'] & \mu^{-1}(0)/G \arrow[d, "h^{\vee}"] \\ & \mathrm{T}^*M \end{tikzcd}$$

Here, the projection π is completely natural and is induced from the projection of the pullback bundle $\pi\colon P^{\#} \to \mathrm{T}^*M$. The map $h^{\vee}$ is the dual of the horizontal lift $h\colon \mathrm{T}M \to \mathrm{T}P$, completely determined and equivalent to the choice of connection. Therefore, we have two different ways to understand Wong's equations of motion. In the space $P^{\#} \times_G Q$, the Hamiltonian function does not get modified but the symplectic structure absorbs the additional factor Ω_η. In the universal model $\mu^{-1}(0)/G$, the symplectic form is canonical but the Hamiltonian function gets twisted by the pullback under $h^{\vee}$.

4.2 The singular minimal coupling

The minimal coupling enables the study of classical particles interacting with Yang–Mills field in the symplectic formulation. The extension of this construction to include systems modeled with E-manifolds was proposed by Mir, Miranda, and Nicolás [6]. More concretely, one of the results proved is the following analogous statement to Theorem 4.3.

Theorem 4.4 (Mir–Miranda–Nicolás [6]). *Let $\pi\colon P \to M$ be a principal G-bundle over an E-manifold $E \to M$ and let (Q, Ω) be a Hamiltonian G-space with moment map μ_Q. Let $P^{\#}$ be the pullback bundle of π by the submersion $E^*_M \to M$.*

Then, the space $E^*_P \times Q$ is a G-Hamiltonian space for the diagonal G-action with moment map $\mu = \mu_P + \mu_Q$. Any choice of connection in P induces a diffeomorphism $\mu^{-1}(0) \simeq P^{\#} \times Q$ which, furthermore, induces a diffeomorphism of the symplectic spaces $\mu^{-1}(0)/G$ and $P^{\#} \times_G Q$.

Throughout the rest of the section we will fix an E-manifold $E_M \to M$ and a principal G-bundle $P \to M$. The proof of Theorem 4.4 follows essentially the same argument as Weinstein. The complication lies in developing the machinery necessary to state and follow the original proof. Since we would like to extend the singularities of our configuration space M to the bundles E_M and P, we need a procedure to do so. The fundamental notion is that of *prolongation*, which dates back at least to the works of de León, Marrero, and Martínez [1].

Definition 4.5. Assume $f\colon N \to M$ is a surjective submersion over a Lie algebroid $\mathcal{A} \to M$. The *prolongation* of A along f, written $\mathcal{L}^f\mathcal{A}$, is the pullback bundle of the morphisms $\mathrm{d}f\colon \mathrm{T}N \to \mathrm{T}M$ and $\rho\colon \mathcal{A} \to M$. As a set, it can be identified with

$$\mathcal{L}^f\mathcal{A} = \{(X, Y) \in \mathcal{A} \times \mathrm{T}N \mid \rho(X) = \mathrm{d}f(Y)\}.$$

Throughout the rest of the section, we fix an E-manifold $E \to M$. The prolongation of the dual bundle $E^*_M \to M$, which can be thought as the singular tangent bundle of the cotangent bundle, carries a natural Liouville form whose differential is symplectic [1]. Thus, the prolongation of E^*_M is a symplectic E-manifold, in strong resemblance to the cotangent bundle of a smooth manifold.

Similarly, we consider the prolongation of the principal G-bundle $P \to M$. Because $E_P \to P$ has a natural action of the Lie group G and the anchor map is injective on an open and dense subset, the action on P lifts to an action on E_P which factors through the standard tangent map. By duality, the action lifts to the dual bundle E_P^* and, moreover, it becomes Hamiltonian with respect to the canonical symplectic structure. The fact that the action of the Lie group G automatically lifts to E_P is only valid for E-manifolds. If we want to establish similar results for symplectic Lie algebroids, stronger compatibility assumptions are needed to define Hamiltonian group actions (see [3]).

The last ingredient in the proof of Theorem 4.4 is the notion of symplectic reduction in the singular setting. The authors rely on a version of the reduction theorem developed by Marrero, Padrón, and Rodríguez-Olmos for symplectic Lie algebroids (Theorem 3.11 in [3]).

In [6] the authors consider some standard configuration spaces with singularities, such as the compactification of a stationary black hole or a general b^m-manifold, motivated by previous contributions in the literature of celestial mechanics. Moreover, they explicitly compute Wong's equations describing the motion of a charged particle interacting with a Yang–Mills field.

5. Conclusions

Symplectic manifolds are fundamental objects in the geometric formulation of Hamiltonian dynamics. These give rise to Poisson brackets, which measure the evolution of observables along the trajectories of the system but are vastly more general. E-Symplectic manifolds lie between both worlds: even though they define Poisson structures, their behavior is closer to symplectic forms. Moreover, they naturally encode certain physical systems with constrained dynamics.

Theorem 4.4 extends the classical minimal coupling procedure to E-symplectic manifolds. In more physical terms, it provides a Hamiltonian formulation of the equations of motion of particles under the interaction with a Yang–Mills field for constrained systems. This result could open the door to study the dynamics of such physical systems using geometric techniques. Indeed, in [7] the authors obtain a b^3-symplectic structure in the planar, restricted, circular three-body problem and, using a contact analogue of the theory described here, discuss the existence of periodic orbits at infinity. No analogue result has been established for charged particles.

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