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Algebraic topology of finite topological spaces

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Resum *(CAT)*
Aquest treball segueix les línies d'estudi de R. Stong i M. McCord dels espais topològics finits. Tot i que són dues aproximacions diferents, tenen un punt de contacte: els posets. Per una banda, classificarem els espais topològics finits a través dels posets, segons el teorema de classificació de Stong. Per l'altra, veurem com els posets codifiquen tant la informació homotòpica d'un poliedre com la d'un espai topològic finit, seguint el teorema de McCord. Conclourem el treball donant un model finit d'una superfície compacta connexa.

Abstract *(ENG)*
This work follows R. Stong and M. McCord's study lines on finite topological spaces. Although they are two different approaches, they intersect at one point: posets. On the one hand, we will classify finite topological spaces through posets, according to Stong's Classification Theorem. On the other hand, following McCord's Theorem, we will examine how posets encode the homotopic information of both polyhedra and finite topological spaces. We will conclude by providing a finite model of a compact connected surface.

Keywords: *finite topological spaces, partially ordered sets (posets), Hasse diagrams, homotopy theory, minimal spaces, simplicial complexes.*
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1. Introduction

A finite topological space is a topological space that has a finite number of points. At first, one might think that these topological spaces are not very interesting, cannot generate many topologies, and that the homotopy groups vanish immediately. However, they have more structure since every topological space can be associated with a partial order. Partially ordered sets, posets for short, are potent combinatorial objects for encoding information about spaces. These, in turn, are related to simplicial complexes.

Michael McCord and Robert Stong were two American mathematicians from the second half of the 20th century who studied finite topological spaces almost simultaneously in 1966, but from two different perspectives. In this work, we study finite topological spaces following these two classical approaches, to see how they relate to each other through posets. Stong's Classification Theorem [6] is based on the internal structure of the spaces, the order structure. On the other hand, McCord's Theorem [4] compares finite spaces with simplicial complexes through homotopy theory, from a more external point of view. Since this document is intended for general audiences, it does not include proofs. However, if the reader is curious, they can consult the references for details.

2. Preliminaries

In this section, we will see that finite topological spaces and finite posets are essentially the same. We will look at basic definitions for working with finite topological spaces and review some properties.

Definition 2.1. A *finite topological space* (X, τ) is a topological space over a finite set of points.

One might think that in a finite set there is a finite number of topologies and that this fact would suffice for classification. However, the notion of homeomorphism is too restrictive, and we aim to understand these spaces using their topological properties concerning homotopy properties. To grasp some concepts, it is recommended to have a basic understanding of homotopy theory.

2.1 Properties

The Alexandroff topology [1] is characterized by the property that the intersection of any family of open sets is open. Finite topological spaces exemplify this topology, since the arbitrary intersection of open sets cannot be infinite. Consequently, we can talk about the smallest open set containing a point, in the sense of an open closure.

Definition 2.2. Given a point x in a finite topological space (X, τ), we define the *minimal open set of* x as the intersection of all open sets containing x:

$$U_x = \bigcap_{x \in \mathcal{O} \in \tau} \mathcal{O}.$$

The minimal open sets form a basis for the topology of X, called the *minimal basis of* X.

Definition 2.3. A *preorder* is a reflexive and transitive relation. A *preordered set* or *preset* is a set with a preorder. A *partial order* is a reflexive, transitive and antisymmetric relation. A *partially ordered set* or *poset* is a set with a partial order.

Proposition 2.4. *Let X be a finite topological space. The binary relation $\leq$ on X defined by the following expression is a preorder:*

$$x \leq y \text{ if and only if } x \in U_y.$$

We have just seen that finite spaces induce a preorder, given by the minimal basis. Now we will see that, in fact, a preorder also induces a topology on a finite set.

Definition 2.5. Given P a preset and $x \in P$, we write $P_{\leq x} := \{z \in P \mid z \leq x\}$. Similarly, $P_{\geq}$, $P_{<}$, and $P_{>}$ are defined.

In the case of a finite topological space X with associated preorder $\leq$, U_x corresponds to $P_{\leq x}$.

Definition 2.6. Let P be a finite preset. The *Alexandroff topology* is the topology defined by the basis

$$\{P_{\leq x} \subseteq P \mid x \in P\}.$$

In fact, these two are equivalent.

Proposition 2.7 ([7, Proposition 2.1.7])**.** *Let $(X, \leq)$ be a preset and τ the Alexandroff topology. Let $\leq'$ be the preorder on X given by the minimal open sets of (X, τ). Then, the two presets $(X, \leq)$ and $(X, \leq')$ coincide.*

Proposition 2.8 ([2, Proposition 1.2.1])**.** *A function $f : X \to Y$ between finite spaces is continuous if and only if it is order-preserving.*

Definition 2.9. A topological space X satisfies the *separation axiom* T_0 if, given two distinct points, there is an open set containing one of them but not the other.

Proposition 2.10 ([7, Proposition 2.1.9])**.** *A finite topological space X is T_0 if and only if its associated preordered set is antisymmetric; therefore, it is a poset.*

We have seen that the correspondence between finite topological spaces and preorders is bijective, and, in fact, if the topological space is T_0, we have antisymmetry and hence a partially ordered set. One of the main consequences of this correspondence is the visual representation that arises: Hasse diagrams.

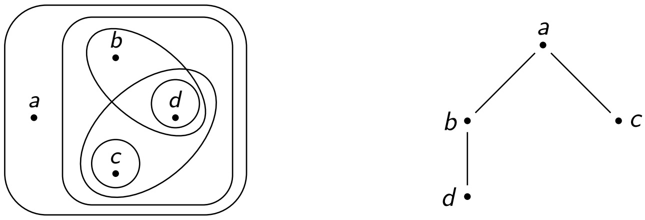
Definition 2.11. The *Hasse diagram* of a poset X is a directed graph whose vertices are the points of X and whose edges are the ordered pairs (x, y) such that $x < y$ and there exists no $z \in X$ such that $x < z < y$. Additionally, the elements are arranged in descending order, with bigger elements in the upper part of the diagram, while smaller ones are placed below.

Given any finite topological space, we can construct a T_0 space that is homotopy equivalent to the given one, by identifying points with the same closure (see [2, Proposition 1.3.1]). We will now study finite topological spaces equivalent under homotopies, therefore, without loss of generality, we can reduce the study to spaces that are T_0 and, hence, posets.

At this point, we discuss how to convert a finite topological space into the Hasse diagram of a poset.

Example 2.12. Let $X = \{a, b, c, d\}$ with the following open sets: $\emptyset$, $\{a, b, c, d\}$, $\{c\}$, $\{d\}$, $\{b, d\}$, $\{c, d\}$ and $\{b, c, d\}$, represented by the interiors of the closed curves of Figure 1(a).

Since X is T_0, it is a poset, thus we can talk about the associated Hasse diagram of X. Let's see how it is constructed. We start with the points corresponding to open sets and place them at the bottommost positions. We can compute the open sets U_x for each $x \in X$: $U_a = \{a, b, c, d\}$, $U_b = \{a, b, c, d\} \cap \{b, d\} \cap \{b, c, d\} = \{b, d\}$, $U_c = \{a, b, c, d\} \cap \{c\} \cap \{c, d\} \cap \{b, c, d\} = \{c\}$ and analogously $U_d = \{d\}$. With this, we establish the order relation: $c < a$, but c is not comparable with b or d; $d < a$ and $d < b$, but since $b < a$, we have the chain $d < b < a$.



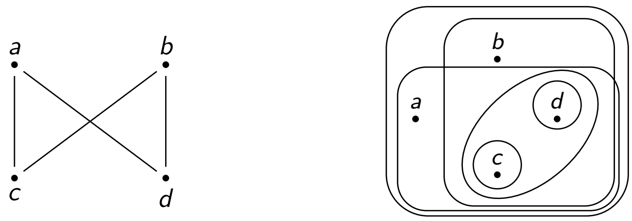
(a) (b)

Figure 1: (a) Open sets of X. (b) Hasse diagram of X.

Now let's see how to obtain a topological space given a the Hasse diagram of a poset.

Example 2.13. Let Y be the poset given by the Hasse diagram seen in Figure 2(a), we want to compute its open sets. Following Definition 2.6, we move through the Hasse diagram starting from the bottom and moving upwards. The sets $\{c\}$ and $\{d\}$ are open.

Now, consider an open set U such that $a \in U$. Since c and d are smaller than a, they must also be in U. Thus, we have the open set $U_a = \{a, c, d\}$. By following a similar process starting from b, we obtain the open set $\{b, c, d\}$. What we have done can be described as "placing our finger" on the point a and descending through all possible edges until reaching the bottom. It is important not to miss any edges, for example, $\{b, d\}$ is not an open set. Therefore, the open sets of X are: $\{c\}$, $\{d\}$, $\{a, c, d\}$, $\{b, c, d\}$, and the unions $\{c, d\}$, $\{a, b, c, d\}$; see Figure 2(b).



(a) (b)

Figure 2: (a) Hasse diagram of space Y. (b) Open sets of Y.

2.2 Bijective correspondence

Finite spaces induce a preorder, and a preorder induces a topology. That is to say, there is a bijective correspondence between finite topological spaces and preorders. Thus, we can talk about the topology of a preorder or the order in a topological space. Table 1 shows a summary of how some properties transfer between finite topological spaces and finite preorders. Some are not explained here, but for further information and details, see the complete final thesis [7].

Finite Topological Space X	Finite Preorder P
U_x	$P_{\leq x}$
$y \in U_x$	$y \leq x$
T_0	Antisymmetric preorder: poset
Diagram of open sets	Hasse diagram
Open set	Down-set
f continuous	f order-preserving
Path, path connected	Fence, order-connected
Homotopy: $f \simeq g$	Fence of maps: $f = f_0 \leq f_1 \geq f_2 \leq \cdots f_n = g$

Table 1: Correspondence between finite topological spaces and finite preorders (posets).

3. Stong's Classification Theorem

This section explores how Stong uses homotopy theory to classify finite topological spaces.

3.1 Minimal spaces: the core

We begin by identifying the smallest space that preserves the homotopy properties of a given finite topological space.

Definition 3.1. Let $x, y \in X$ be two points in a finite topological space. We say that x *covers* y if $x > y$ and for all $z \in X$ such that $x > z \geq y$, we have $z = y$. It can also be said that y *is covered* by x.

Definition 3.2. Let X be a finite T_0 topological space. A point $x \in X$ is called a *down beat point* if it covers one and only one element of X. Dually, x is an *up beat point* if it is covered by exactly one element. Points that satisfy either of these properties are referred to as *beat points* of X.

Remark 3.3. In the Hasse diagram, x is a down beat point if it has exactly one lower edge. In the topological space, this is equivalent to saying that the set $\hat{U}_x = U_x \setminus \{x\}$ has a maximum. Similarly, x is an up beat point if it has exactly one upper edge in the Hasse diagram.

We can see in Example 2.12 that b, d and c are up beat points, b is also a down beat point and a is neither of them. There are no beat points in Example 2.13.

Definition 3.4. A finite T_0 topological space X is *minimal* if it has no beat points. The *core* of a finite topological space X is a subspace that is also minimal as a topological space.

Given a finite topological space X, its core can be constructed by removing beat points one at a time. This process preserves the homotopy properties of X because the resulting subspace is a strong deformation retract (see [2, Proposition 1.3.4]).Observe that this minimal subspace always exists. If X has no beat points, it is already minimal, making X its own core, as illustrated by the space in Example 2.13. If beat points are present, they can be removed successively until a minimal space is obtained. For instance, in Example 2.12, we can retract d to b, then b to a, and lastly c to a; therefore, $\{a\}$ is the core of X. As you may have deduced, the space X is homotopic to a point; therefore, X is indeed contractible, which can be easily observed in the Hasse diagram, rather than in the description of X by its open sets.

Note that minimal is in the sense of not having beat points, not of having a few points. This will be the smallest subspace of a finite topological space that keeps the original homotopy properties. The space of Example 2.12 is not minimal, whereas the one in Example 2.13 is minimal.

Example 3.5. Let X be the finite topological space associated with the Hasse diagram shown in Figure 3(a). We compute its core by removing the beat points. First, since b is an up beat point, we retract it towards a. Then, we retract c towards a because it is an up beat point of $X \setminus \{b\}$. Finally, we retract e towards a because it is an up beat point of $X \setminus \{b, c\}$. The resulting subspace $X \setminus \{b, c, e\}$ is minimal and therefore is the core of X. Note that changing the order of this process leads to the same result, bearing in mind that we can only retract one beat point at a time.

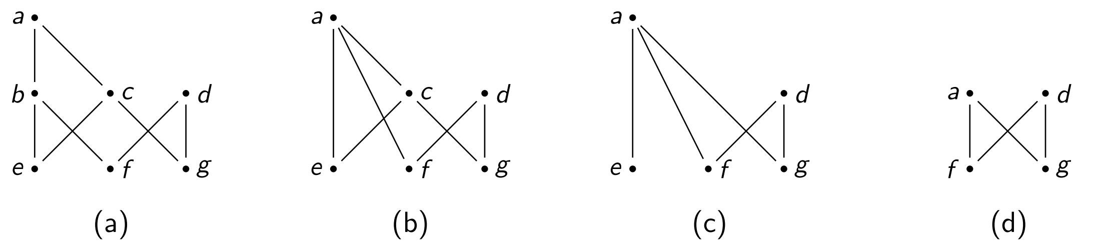


Figure 3: (a) Space X not minimal. (b) $X \setminus \{b\}$. (c) $X \setminus \{b, c\}$. (d) $X \setminus \{b, c, e\}$, the core of X.

3.2 The Theorem

In his work, Stong [6] introduces a matricial approach to classify finite spaces. However, in this paper we will not adopt Stong's method. Instead, the Classification Theorem can be proven using more straightforward propositions, as discussed in [7, Corollary 2.3.10] or [2, Corollary 1.3.7].

Theorem 3.6 (Classification Theorem (Stong))**.** *A homotopy equivalence between minimal finite topological spaces is a homeomorphism. In particular, the core of a finite space is unique up to homeomorphism, and two finite topological spaces are homotopy equivalent if and only if they have homeomorphic cores.*

The crucial point is that with minimality, we can compare spaces by homeomorphism instead of homotopy. Essentially, finite topological spaces are determined up to homeomorphism by their core, there is a *bijection* between posets.

Example 3.7. Consider the following finite T_0 topological spaces X and Y given by Figure 4(a) and (c). They are very similar, but are they homotopic? We already computed the core of X in Example 3.5. For Y, we just retract a, that is a down beat point, towards b and we have the cores shown in Figure 4(c) and (d).

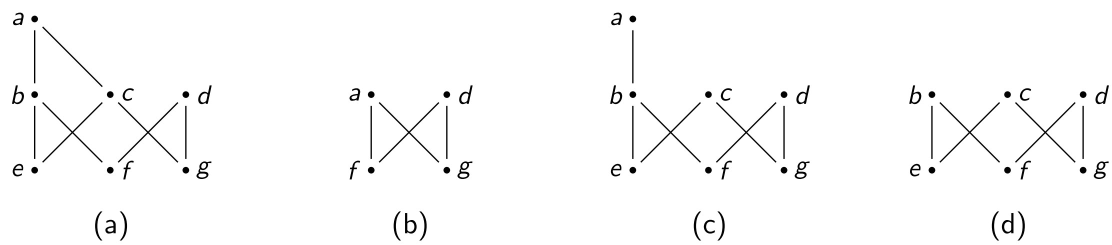


Figure 4: (a) Space X. (b) The core of X. (c) Space Y. (d) The core of Y.

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The cores of X and Y are not homeomorphic. Therefore, by Stong's Theorem 3.6, X and Y are not homotopy equivalent. But these two spaces are not visually much different, is there any other way to compare them? The next section is dedicated to relaxing the equivalence criteria to see that they aren't that different.

4. McCord's Theorem

McCord studies finite topological spaces by associating them with abstract combinatorial objects, known as *simplicial complexes*. These complexes can be assigned with a topological representation, called the *geometric realization*, which enables their visualization as geometric structures in Euclidean space, connected coherently, forming geometric shapes such as triangles and tetrahedra. If the reader is unfamiliar with these concepts, they can refer to any introduction to simplicial complexes, such as [5], [2, Appendix] or [7].

4.1 Simplicial complexes

Definition 4.1. A *simplicial complex* $\mathcal{K}$ consists of a set $V_{\mathcal{K}}$, called the set of vertices, and a set $S_{\mathcal{K}}$ of finite nonempty subsets of $V_{\mathcal{K}}$, which is called the set of simplices, satisfying that any subset of $V_{\mathcal{K}}$ of cardinality one is a simplex and any nonempty subset of a simplex is a simplex.

Definition 4.2. The *geometric realization* $|\mathcal{K}|$ of a simplicial complex $\mathcal{K}$ is the set of formal convex combinations $\sum_{v\in\mathcal{K}} \alpha_v v$ such that $\{v|\alpha_v > 0\}$ is a simplex of $\mathcal{K}$.

Definition 4.3. Let X be a T_0 finite topological space. The *simplicial complex associated to* X, or *order complex*, denoted by $\mathcal{K}(X)$, is the simplicial complex whose n-simplices are chains of length n:

$$x_0 < x_1 < \cdots < x_n,$$

where the order relation is given by Proposition 2.4.

Example 4.4. Let X be the finite T_0 topological represented in Figure 5(a). Let's see how to construct the associated simplicial complex $\mathcal{K}(X)$. First, the elements of X are the vertices or 0-simplices. Next, we look at the longest chains. Here we have $\{d < b < a\}$ and $\{e < b < a\}$. Therefore, we have two 2-simplices. Note that they share an edge, $\{b < a\}$.

Next, we go down in dimension. In this case, we need to add an edge from c to d and another from c to e, because the other chains of size 1 are already represented as edges (1-simplices) of the 2-simplices. Finally, we graphically represent the geometric realization of $\mathcal{K}(X)$ in Figure 5(b).

Figure 5: (a) Space X. (b) Order complex $\mathcal{K}(X)$.

The inverse process can be done by constructing the face poset of $\mathcal{K}$, obtaining a topological space. The face poset is obtained by taking the simplices of a given simplicial complex $\mathcal{K}$ as elements and defining the order relation by inclusion. Note that this would not result in the original space but in a homotopic one.

4.2 The McCord map

Definition 4.5. Given $\mathcal{K}$ and $\mathcal{L}$ two simplicial complexes, a simplicial map $\varphi\colon \mathcal{K} \to \mathcal{L}$ is a vertex map $\varphi'\colon V_{\mathcal{K}} \to V_{\mathcal{L}}$ that sends simplices into simplices.

Note that when defining a map between simplicial complexes, it suffices to specify the map on the vertices, provided that the vertex mapping respects the combinatorial structure of the image complex. This follows from the fact that each simplex in a complex is uniquely determined by its vertices; therefore, once the map is defined on the vertices, it extends naturally and consistently to all simplices.

Definition 4.6. Let X and Y be finite T_0 topological spaces, and $f\colon X \to Y$ a continuous map. Then, the *associated simplicial map* $\mathcal{K}(f)\colon \mathcal{K}(X) \to \mathcal{K}(Y)$ is defined by $\mathcal{K}(f)(x) = f(x)$.

McCord [4, Theorem 6] proved that given a continuous map that is locally a weak homotopy equivalence over a basis-like open cover, then it is globally so. We will use this theorem, the previous definition and the minimal basis to show that the two spaces we have seen before are weak homotopy equivalents.

Example 4.7. As previously discussed in Example 3.5, no homotopy equivalence exists between X and Y because their cores are not homeomorphic. But using the following map, we can prove that there is a weak homotopy equivalence between X and Y. Consider X' and Y' the cores of X and Y, respectively. The map $f\colon Y' \to X'$ is given by $f(a_1) = f(a_2) = f(a_3) = a$, $f(b) = b$, $f(c) = c$, $f(d) = d$; see Figure 6.

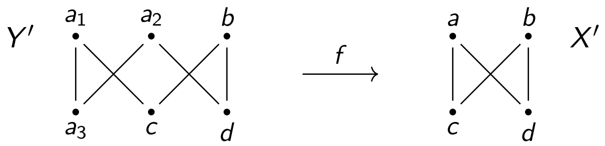

Figure 6: Map f.

It is shown that f is order-preserving, and therefore continuous. Then, by the fact that preimages of minimal open sets are contractible [7, Corollary 2.2.12] and [4, Theorem 6], we obtain that f is a weak homotopy equivalence.

Every homotopy equivalence induces an isomorphism on the homotopy groups, but two spaces with isomorphic homotopy groups may not be homotopy equivalent. This correspondence is called a *weak homotopy equivalence*. This means that spaces X and Y have isomorphic homotopy groups, and with this relaxed criteria, we can finally say that they are equivalent. In fact, they both are finite models of the sphere S^1.

Observe that, given a finite T_0 topological space X and its geometric realization $|\mathcal{K}(X)|$, any point $\alpha \in |\mathcal{K}(X)|$ can be expressed, by construction, in terms of coordinates over a chain $x_1 < x_2 < \cdots < x_n$ in X in the form $\alpha = \sum_{i=1}^{n} \lambda_i x_i$, where $\lambda_i > 0$ for all $1 \leq i \leq n$ and $\sum_{i=1}^{n} \lambda_i = 1$. The *support* of α is precisely this chain: $\mathrm{supp}(\alpha) = \{x_1, x_2, \dots, x_n\}$.

Definition 4.8. Let X be a finite T_0 topological space and $\alpha \in |\mathcal{K}(X)|$ a point in the geometric realization of the simplicial complex associated with X such that $\text{supp}(\alpha) = \{x_1, x_2, \dots, x_n\} \subseteq X$. The *McCord map* is the map $\mu_X : |\mathcal{K}(X)| \to X$ defined by

$$\mu_X(\alpha) = \min(\text{supp}(\alpha)) = x_1.$$

Example 4.9. We start from Example 4.4, where we computed the simplicial complex associated with a topological space X. A point $\alpha \in |\mathcal{K}(X)|$ in the interior of the triangle abe can be written as $\alpha = \lambda_1 e + \lambda_2 b + \lambda_3 a$ with $\sum_{i=1}^{3} \lambda_i = 1$. Then, $\mu_X(\alpha) = \min(\{e, b, a\}) = e$, and therefore every point in the interior of this triangle will map to e, represented in green in Figure 7. Similarly, points in the interior of adb will map to d, represented in red.

Now consider a point on the interior of the edge ae. Since the support is $\{e, a\}$, the minimum is e. More generally, the smallest vertex, concerning the poset order, of all the ones contained in the simplex, will "absorb" the points in the interior of the simplex. Figure 7 shows the McCord map, where colours represent the preimages of the vertices.

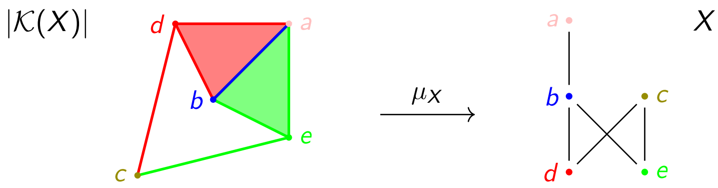

Figure 7: McCord map coloured.

From here, the objective is to conclude this section by establishing that for any given polyhedron, there exists a corresponding finite topological space, and conversely, for any finite topological space, there exists a polyhedron that models it. First, we prove that this correspondence is a weak homotopy, followed by its application in proving McCord's Theorem. The proofs provided by Barmak [2, Theorem 1.4.6] and McCord [4, Theorem 1] are somewhat intricate or may lack comprehensive details. For a clearer comprehension of this demonstration, you can refer to [7, Theorem 3.1.8].

Theorem 4.10 (McCord [4, Theorem 1])**.**

1. *For every finite topological space X, there exists a finite simplicial complex $\mathcal{K}$ and a weak homotopy equivalence $f : |\mathcal{K}| \to X$.*
2. *For every finite simplicial complex $\mathcal{K}$, there exists a finite topological space X and a weak homotopy equivalence $f : |\mathcal{K}| \to X$.*

This theorem states that every finite topological space has an associated simplicial complex that preserves properties up to weak homotopy equivalence and vice versa, that every simplicial complex has a finite topological space that is weakly homotopy equivalent. Both finite topological spaces and simplicial complexes have a strong combinatorial structure, which is convenient to work with. This is useful because it allows us to study non-finite topological spaces algorithmically through finite ones, by using triangulations of spaces such as the sphere.

Recall that a triangulation is a homeomorphism between a topological space and a simplicial complex. For example, a triangle is homeomorphic to S^1, or a hollow tetrahedron is homeomorphic to S^2, and these are simplicial complexes.

4.3 Finite models of non-finite spaces

We conclude this work by giving some examples of use to model the compact connected surfaces. Due to space constraints, we will not see the 2-torus in this paper; however, it can be found in [7, pp. 32–34]. Beginning with the sphere S^2, consider the following triangulation h given by a hollow tetrahedron; see Figure 8.

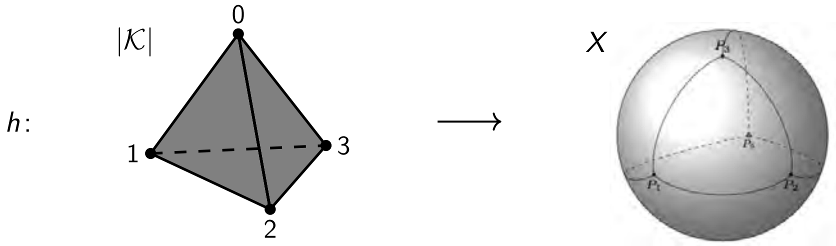

Figure 8: Triangulation of a sphere. Sphere is extracted from [3].

Each vertex is labelled with a number, and each simplex is labelled with the numbers of its vertices. We then construct the face poset of $\mathcal{K}$ and obtain the Hasse diagram of the associated poset; see Figure 9. Note that this gives us a *minimal finite* topological space that models the non-finite space S^2.

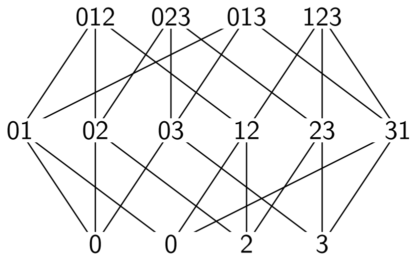

Figure 9: Finite model of S^2.

Also, note that a triangulation with additional vertices would result in a larger finite topological space, although it would also be minimal. This highlights the fact that minimal refers to not having beat points, not to having a few points. Despite being weakly homotopy equivalent, these spaces wouldn't be homotopy equivalent, since they wouldn't be homeomorphic. This underlines why we need to relax the equivalence criteria because we know that they both model the same surface, therefore they must be equivalent in some context.

The next example is the projective plane RP^2. Consider the triangulation illustrated in Figure 10. The Hasse diagram of its associated face poset can be seen in Figure 11.

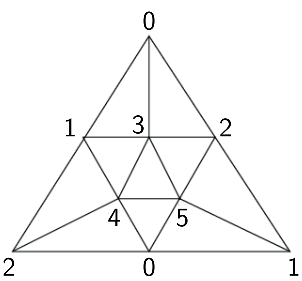

Figure 10: Triangulation of RP^2.

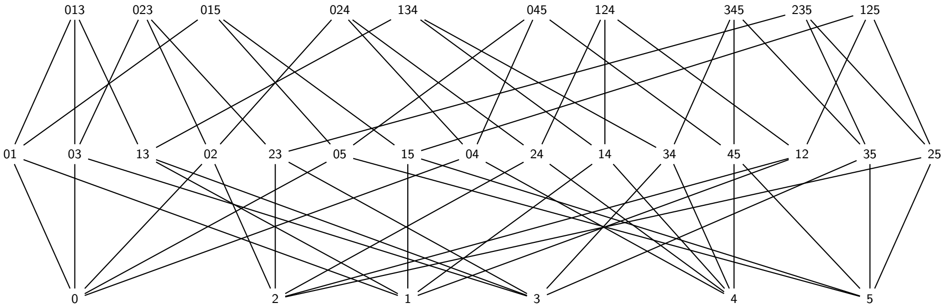

Figure 11: Finite model of RP^2.

The Classification Theorem for compact connected surfaces states that every compact connected surface can be described in terms of spheres, tori, projective planes, and connected sums of these. As a final point, we will provide an overview of how to compute the connected sum, allowing us to model all compact connected surfaces by using the previously provided models and incorporating this procedure.

Consider two spheres S^2 triangulated as in Figure 8. The connected sum consists of identifying vertices and corresponing edges, and eliminating the faces comprised within these vertices; see Figure 12. We proceed as follows: identify vertex 0 of the first sphere with vertex 0 of the second (denoted as $0'$), vertex 2 of the first with vertex 1 of the second (denoted as 4), and vertex 3 of the first with vertex 2 of the second (denoted as $2'$), while keeping vertices 1 of the first and 3 of the second unchanged, identify the corresponding edges, and finally remove the interior triangle formed by $0'$, $2'$ and 4.

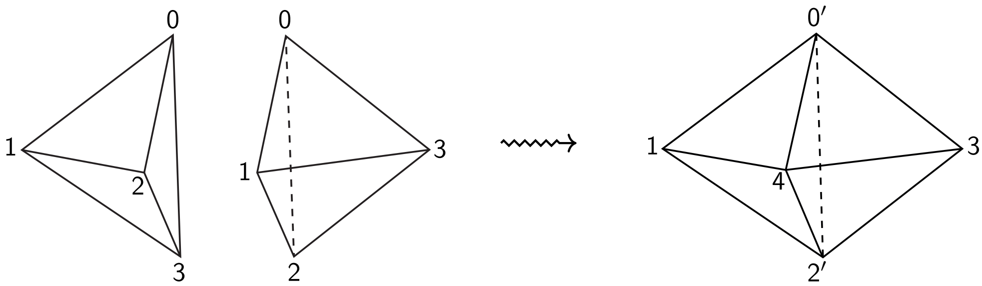

Figure 12: Diagram of the connected sum of two spheres S^2.

Given two associated posets of the spheres, as in Figure 9, we construct the associated poset of the connected sum by following the mentioned procedure: identifying corresponding vertices and omitting the identified triangle; see Figure 13. This approach avoids the need to calculate the poset for the resulting connected sum of spheres.

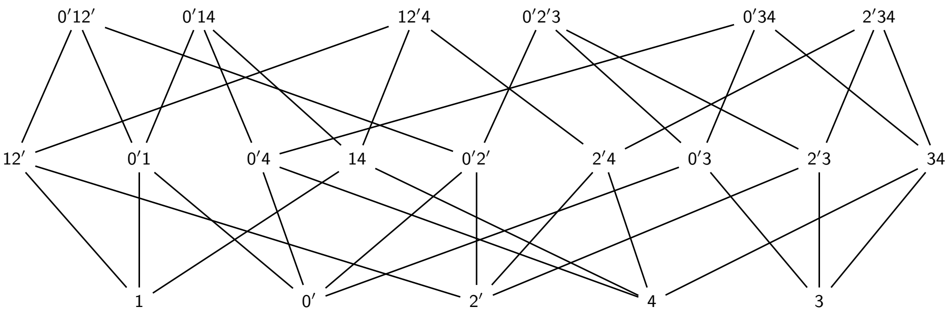

Figure 13: Finite model of a connected sum of two spheres S^2.

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