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Properties of triangular partitions and their generalizations

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Resum *(CAT)*
Una partició entera es diu triangular si el seu diagrama de Ferrers es pot separar del seu complement (com a subconjunt de $\mathbb{N}^2$) amb una línia recta. Aquest article es basa en alguns desenvolupaments recents sobre el tema per derivar noves propietats enumeratives, geomètriques i algorísmiques d'aquests objectes. La investigació s'estén després a generalitzacions en dimensions superiors, anomenades particions piramidals, i a particions convexes i còncaves, definides com particions amb un diagrama de Ferrers que pot ser separat del seu complement per una corba convexa o còncava.

Abstract *(ENG)*
An integer partition is said to be triangular if its Ferrers diagram can be separated from its complement (as a subset of $\mathbb{N}^2$) by a straight line. This article builds on some recent developments on the topic in order to derive new enumerative, geometric and algorithmic properties of these objects. The research is then extended to higher-dimensional generalizations, called pyramidal partitions, and to convex and concave partitions, defined as partitions whose Ferrers diagram can be separated from its complement by a convex or concave curve.

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1. Introduction

An integer partition is said to be triangular if its Ferrers diagram can be separated from its complement by a straight line. These objects first appeared in the contexts of combinatorial number theory [3] and computer vision [4]. From a combinatorial perspective, they were first studied by Onn and Sturmfels [11], who defined them in any dimension and called them *corner cuts*. Shortly after, Corteel et al. [5] obtained the generating function for the number of 2-dimensional corner cuts. More recently, triangular partitions have attracted interest in the field of algebraic combinatorics. Motivated by work of Blasiak et al. [2] generalizing the shuffle theorem for paths under a line, Bergeron and Mazin [1] coined the term *triangular partitions* and studied some of their combinatorial properties.

In this article we present new enumerative, geometric and algorithmic properties of triangular partitions and their generalizations. In Section 2 we give basic definitions and some results from [1, 5]. In Section 3 we introduce a natural alternative characterization of triangular partitions, as those such that the convex hull of the Ferrers diagram and that of its complement do not intersect. Moreover, we characterize which points may be added to or removed from the Ferrers diagram while preserving triangularity.

In Section 4, we present two ways to encode triangular partitions in terms of balanced words, and use one of them to implement an algorithm which, for a given N, computes the number of triangular partitions of size $n \leq N$ in time $\mathcal{O}(N^{5/2})$. This allows us to obtain the first 10^5 terms of this sequence, while just 39 terms were known previously.

In Section 5, refining the approach from [5], we obtain generating functions for triangular partitions with a given number of removable and addable cells. In Section 6, we present a recurrence for the number of triangular partitions contained in a fixed triangular partition, as well as an explicit formula involving Euler's totient function for the case where the fixed partition is a staircase. A new combinatorial proof of Lipatov's enumeration theorem for balanced words [8] is obtained as a byproduct.

Section 7 studies pyramidal partitions, which are an extension of triangular partitions to higher dimensions. We prove that the characterization in terms of convex hulls generalizes nicely and that, for dimension 3 or higher, the number of removable and addable cells can be arbitrarily large. We also describe the residue modulo d of the number of d-dimensional pyramidal partitions of size n, for d prime.

In Section 8, convex and concave partitions are analyzed. These are partitions whose Ferrers diagram can be separated from its complement by a convex or concave line. We present several characterizations and we describe their removable and addable cells in terms of convex hulls. Finally, we prove that there exist constants a, b, c such that the number of convex partitions of size n is greater than $\exp(a\sqrt[3]{n})$ and smaller than $\exp(b\sqrt[3]{n}\log n)$, and the number of concave partitions of size n is greater than $\exp(c\sqrt[3]{n})$.

Due to space constraints, proofs are omitted from this article. A more thorough explanation of the results is detailed by Elizalde and the present author in [7].

2. Background

A *partition* λ is a weakly decreasing sequence of positive integers, called the *parts* of λ. We will denote $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$, or $\lambda = \lambda_1\lambda_2\dots\lambda_k$ when there is no possibility of confusion. We call $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_k$ the *size* of λ. If $|\lambda| = n$, we say that λ is a partition of n.

Let $\mathbb{N}$ denote the set of positive integers. The *Ferrers diagram* of λ is the set of lattice points

$$\{(a,b) \in \mathbb{N}^2 \mid 1 \le b \le k,\ 1 \le a \le \lambda_b\}.$$

We will often identify a lattice point (a,b) with the unit square (called a *cell*) whose north-east corner is (a,b). In particular, we say that a cell lies above, below or on a line when the north-east corner does. The Ferrers diagram can then be interpreted as a set of cells. We will often identify λ with its Ferrers diagram, and use notation such as $c = (a,b) \in \lambda$.

Let $\sigma^k = (k, k-1, \dots, 2, 1)$ denote the *staircase partition* of k parts. The *conjugate* λ' of λ is obtained by reflecting its Ferrers diagram about the $y = x$ axis. The *complement* of λ is defined to be the set $\mathbb{N}^2 \setminus \lambda$, where λ is identified with its Ferrers diagram.

Definition 2.1. A partition τ is *triangular* if its Ferrers diagram consists of the points in $\mathbb{N}^2$ that lie on or below the line that passes through $(0,s)$ and $(r,0)$ for some $r, s \in \mathbb{R}_{>0}$, called a *cutting line*.

See the left of Figure 1 for an example. We often use τ to denote a triangular partition.

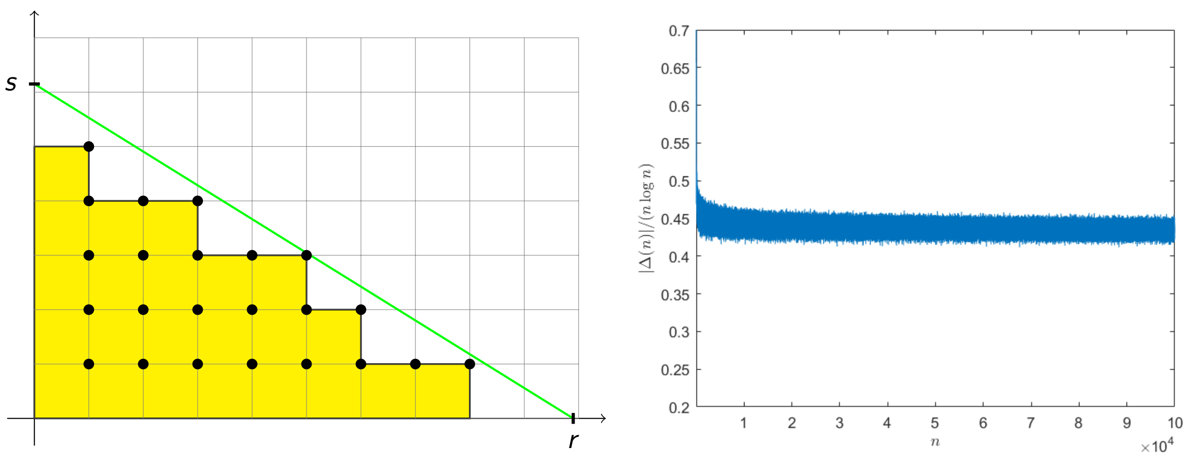

Figure 1: Left: A cutting line for the triangular partition $(8,6,5,3,1)$. Right: The first 10^5 terms of the sequence $|\Delta(n)|/(n \log n)$.

Denote by Δ the set of all triangular partitions and by $\Delta(n)$ the set of triangular partitions of size n. Corteel et al. [5] obtain the generating function of $|\Delta(n)|$ and bound the asymptotic growth of this number.

Theorem 2.2 ([5]). *The generating function for triangular partitions can be expressed as*

$$G_\Delta(z) = \sum_{n\ge 0} |\Delta(n)| z^n = \frac{1}{1-z} + \sum_{\gcd(a,b)=1} \sum_{\substack{0\le j<a \\ 0\le i<b}} \sum_{1\le m<k} z^{N_\Delta(a,b,k,m,i,j)},$$

where

$$N_\Delta(a,b,k,m,i,j) = (k-1)\left(\frac{(a+1)(b+1)}{2} - 1\right) + \binom{k-1}{2} ab + ij \\ + i(k-1)a + j(k-1)b + T(a,b,j) + T(b,a,i) + m, \tag{1}$$

and $T(a,b,j) = \sum_{r=1}^{j} (\lfloor rb/a \rfloor + 1)$.

Theorem 2.3 ([5]). *There exist positive constants c and c' such that, for all $n > 1$,*

$$cn \log n < |\Delta(n)| < c' n \log n.$$

Let $c = (i, j)$ be a cell of a triangular partition $\lambda = \lambda_1 \dots \lambda_k$. Define the *arm length* and the *leg length* of c to be $a(c) = \lambda_j - i$ and $\ell(c) = \lambda'_i - j$, that is, the number of cells to the right of c in its row, and above c in its column, respectively. Bergeron and Mazin [1] characterize triangular partitions and study the number of cells that can be added or removed while preserving triangularity.

Lemma 2.4 ([1, Lemma 1.2]). *A partition λ is triangular if and only if $t^-_\lambda < t^+_\lambda$, where*

$$t^-_\lambda = \max_{c\in\lambda} \frac{\ell(c)}{a(c)+\ell(c)+1}, \quad \textit{and} \quad t^+_\lambda = \min_{c\in\lambda} \frac{\ell(c)+1}{a(c)+\ell(c)+1}.$$

Definition 2.5. A cell of $\tau \in \Delta$ is *removable* if removing it from τ yields a triangular partition. A cell of the complement $\mathbb{N}^2 \setminus \tau$ is *addable* if adding it to τ yields a triangular partition.

Lemma 2.6 ([1, Lemma 4.5]). *Every nonempty triangular partition has either one removable cell and two addable cells, two removable cells and one addable cell, or two removable cells and two addable cells.*

3. Characterization of triangular partitions

In this section, we introduce a new characterization of triangular partitions in terms of convex hulls. This characterization is natural and arguably simpler than the one given in Lemma 2.4 by Bergeron and Mazin [1], which involves the computation of an expression in terms of arm and leg lengths for each cell. We also present a way to identify removable and addable cells. The convex hull of a set $S \subseteq \mathbb{N}^2$ will be denoted by $\mathrm{Conv}(S)$.

Proposition 3.1. *A partition λ is triangular if and only if $\mathrm{Conv}(\lambda) \cap \mathrm{Conv}(\mathbb{N}^2 \setminus \lambda) = \emptyset$.*

We will use the term *vertex* to refer to a 0-dimensional face of a polygon; in particular, not all lattice points of $\mathrm{Conv}(\tau)$ are vertices.

Proposition 3.2. *Two cells in $\tau \in \Delta$ are removable if and only if they are consecutive vertices of $\mathrm{Conv}(\tau)$ and the line passing through them does not intersect $\mathrm{Conv}(\mathbb{N}^2 \setminus \tau)$. Similarly, two cells in $\mathbb{N} \setminus \tau$ are addable if and only if they are consecutive vertices of $\mathrm{Conv}(\mathbb{N}^2 \setminus \tau)$ and the line passing through them does not intersect $\mathrm{Conv}(\tau)$.*

An immediate corollary is that a triangular partition cannot have more than two removable cells and two addable cells, as we know from Lemma 2.6 by Bergeron and Mazin [1].

A similar characterization in terms of convex hulls for a single removable cell is proved by Elizalde and the present author in [7], and is then used to describe an algorithm that determines whether a partition λ of n into k parts is triangular. Said algorithm has complexity $\mathcal{O}(k)$ for the initialization and $\mathcal{O}(\min\{k, \sqrt{n}\})$ for the rest of its steps, whereas an algorithm based on Bergeron and Mazin's Lemma 2.4 would take time $\mathcal{O}(n)$.

4. Bijections to balanced words and efficient generation

In this section, we present two different interpretations of triangular partitions in terms of finite Sturmian words, also known as balanced words. The first interpretation, which is hinted at in [1], is quite natural, and it will allow us to prove an enumeration formula in Section 6. The second one relates each triangular partition to a balanced word together with two positive integers, and it will be used in Section 4.4 to implement an efficient algorithm to count triangular partitions by size.

4.1 Background on balanced words

A finite consecutive subword of a word is called a *factor*. An infinite binary word s is *Sturmian* if, for every $\ell \geq 1$, the number of factors of s of length ℓ is exactly $\ell + 1$. The applications of Sturmian words range from combinatorics and number theory to dynamical systems; see [9] for a thorough study.

A finite binary word $w = w_1 \dots w_\ell$ is a factor of some Sturmian word if and only if it is *balanced*, that is, for any positive integers $h \leq \ell$ and $i, j \leq \ell - h + 1$, we have

$$|(w_i + w_{i+1} + \cdots + w_{i+k-1}) - (w_j + w_{j+1} + \cdots + w_{j+k-1})| \leq 1.$$

This condition states that for any two factors of w of the same length, the number of ones in these factors differs by at most 1. Denote by $\mathcal{B}$ the set of all balanced words, and by $\mathcal{B}_\ell$ the set of those of length ℓ.

The following enumeration formula for balanced words was first proved by Lipatov [8]. Let φ denote Euler's totient function.

Theorem 4.1 ([8])**.** *The number of balanced words of length ℓ is*

$$|\mathcal{B}_\ell| = 1 + \sum_{i=1}^{\ell} (\ell - i + 1)\varphi(i).$$

4.2 First Sturmian interpretation

Definition 4.2. A triangular partition is *wide* if all its parts are distinct. A partition is *tall* if its conjugate is wide.

It can be shown that every triangular partition must be wide or tall, and it is both wide and tall if and only if it is a staircase. The following proposition is a consequence of a well-known bijection between balanced words and lattice paths with steps in $\{(1,0),(1,1)\}$ (see [9]).

Given a wide triangular partition $\tau = \tau_1 \dots \tau_k$, define the binary word

$$\omega(\tau) = 10^{\tau_1-\tau_2-1}10^{\tau_2-\tau_3-1} \dots 10^{\tau_{k-1}-\tau_k-1}10^{\tau_k-1}. \tag{2}$$

Since τ is wide, the exponents are nonnegative. For example, $\omega(86531) = 10110101$.

Proposition 4.3. *For every $k, \ell \geq 1$, the map ω is a bijection between the set of wide triangular partitions with k parts and first part equal to ℓ, and the set of balanced words of length ℓ with k ones that start with 1.*

4.3 Second Sturmian interpretation

To our knowledge, our second encoding of triangular partitions using balanced words is new. Let $\mathcal{W}$ be the set of wide triangular partitions with at least two parts, and let $\mathcal{B}^0$ denote the set of balanced words that contain at least one 0.

First we describe the set of differences of consecutive parts in a wide triangular partition. For $\tau = \tau_1 \dots \tau_k \in \mathcal{W}$, define

$$\mathcal{D}(\tau) = \{\tau_1 - \tau_2, \tau_2 - \tau_3, \dots, \tau_{k-1} - \tau_k\}.$$

Lemma 4.4. *For any $\tau = \tau_1 \dots \tau_k \in \mathcal{W}$, there exists $d \in \mathbb{N}$ such that $\tau_k \leq d + 1$ and either $\mathcal{D}(\tau) = \{d\}$ or $\mathcal{D}(\tau) = \{d, d + 1\}$.*

Define $\min(\tau) = \tau_k$, $\mathrm{dif}(\tau) = \min \mathcal{D}(\tau)$, and $\mathrm{wrd}(\tau) = w_1 \dots w_{k-1}$, where, for $i \in [k - 1]$, we let $w_i = \tau_i - \tau_{i+1} - \mathrm{dif}(\tau)$. Lemma 4.4 guarantees that $\mathrm{wrd}(\tau)$ is a binary word.

Theorem 4.5. *The map $\chi = (\min, \mathrm{dif}, \mathrm{wrd})$ is a bijection between $\mathcal{W}$ and the set*

$$\mathcal{T} = \{(m, d, w) \in \mathbb{N} \times \mathbb{N} \times \mathcal{B}^0 \mid m \leq d + 1;\ w1 \in \mathcal{B}^0 \text{ if } m = d + 1\}.$$

Its inverse is given by the map

$$\xi(m, d, w_1 \dots w_{k-1}) = \tau_1 \dots \tau_k, \quad \text{where } \tau_i = m + \sum_{j=i}^{k-1}(w_j + d) \text{ for } i \in [k].$$

Additionally, given $\tau \in \mathcal{W}$ with image $\chi(\tau) = (m, d, w)$, its number of parts equals the length of w plus one, and its size is

$$|\tau| = km + \binom{k}{2} d + \sum_{i=1}^{k-1} i w_i. \tag{3}$$

4.4 Efficient generation

Before this work, the entry of the OEIS [10, A352882] for the number triangular partitions of n only included values for $n \leq 39$. These are the terms listed in [5], where they are obtained from the generating function in Theorem 2.2. This approach turns out to be impractical for large n.

Theorem 4.5 can be used to implement a much more efficient algorithm that can quickly compute the first 10^5 terms of the sequence. Consider the tree where each vertex is a balanced word of length at most $\lfloor\sqrt{2N}\rfloor$, and the parent of a nonempty word is the word obtained by removing its last letter. On input N, our algorithm runs a depth first search through this tree.

For each $w \in \mathcal{B}_\ell$ with $\ell \leq \sqrt{2N}$, the algorithm finds all the values $m, d \in \mathbb{N}$ such that $(m, d, w) \in \mathcal{T}$, as defined in Theorem 4.5, and such that the size function given in equation (3) is at most N. Each triplet (m, d, w) corresponds to two partitions, the wide triangular partition $\tau = \chi(m, d, w)$ and its conjugate, except when $w = 0^{k-1}$ (for some $k \geq 2$) and $m = d$, in which case it accounts for only one partition, the staircase σ^k.

A C++ implementation of this algorithm can be found at [6]. In a standard laptop computer, this algorithm generates the first 10^3 terms of the sequence $|\Delta(n)|$ in under one second, the first 10^4 terms in under ten seconds, and the first 10^5 terms in under one hour.

Proposition 4.6. *The above algorithm finds $|\Delta(n)|$ for $1 \leq n \leq N$ in time $\mathcal{O}(N^{5/2})$. Additionally, it can be modified to generate all (resp. all wide) triangular partitions of size at most N in time $\mathcal{O}(N^3 \log N)$ (resp. $\mathcal{O}(N^{5/2} \log N)$).*

The plot on the right of Figure 1 portrays the first 10^5 terms of the sequence $|\Delta(n)|/(n \log n)$. A qualitative study suggests that, for large n, this sequence oscillates between two decreasing functions that differ by about 0.05.

5. Generating functions for subsets of triangular partitions

Let Δ_1 and Δ_2 denote the subsets of triangular partitions with one removable cell and with two removable cells, respectively. Let Δ^1 and Δ^2 denote the subsets of triangular partitions with one addable cell and with two addable cells, respectively. Let $\Delta_2^2 = \Delta_2 \cap \Delta^2$. Denote partitions of size n in each subset by $\Delta_1(n)$, $\Delta_2(n)$, $\Delta^1(n)$, $\Delta^2(n)$ and $\Delta_2^2(n)$. In this section we obtain generating functions for each of these sets, refining Theorem 2.2. In the following proposition, $N_\Delta(a, b, k, m, i, j)$ is the function defined in equation (1).

Proposition 5.1. *The generating function for triangular partitions with two removable cells can be expressed as*

$$G_{\Delta_2}(z) = \sum_{n \geq 0} |\Delta_2(n)| z^n = \sum_{\gcd(a,b)=1} \sum_{\substack{0 \leq j < a \\ 0 \leq i < b}} \sum_{k \geq 2} z^{N_\Delta(a,b,k,k,i,j)}.$$

Proposition 5.2. *The generating functions for partitions in Δ_1, Δ^2, Δ^1, Δ_2^2 can be written in terms of $G_\Delta(z)$ (given in Theorem 2.2) and $G_{\Delta_2}(z)$ (given in Proposition 5.1) as follows:*

$$G_{\Delta_1}(z) = G_\Delta(z) - G_{\Delta_2}(z) - 1, \qquad G_{\Delta^2}(z) = \frac{1-z}{z} G_\Delta(z) + \frac{1}{z} G_{\Delta_2}(z) - \frac{1}{z},$$

$$G_{\Delta^1}(z) = \frac{2z-1}{z} G_\Delta(z) - \frac{1}{z} G_{\Delta_2}(z) + \frac{1}{z}, \qquad G_{\Delta_2^2}(z) = \frac{1-2z}{z} G_\Delta(z) + \frac{1+z}{z} G_{\Delta_2}(z) - \frac{1}{z}.$$

We have used Proposition 5.1 in order to implement an algorithm to find $|\Delta_2(n)|$, available at [6]. The initial terms of the sequences $|\Delta_1(n)|$ and $|\Delta_2(n)|$ suggest that $|\Delta_2(n)| > |\Delta_1(n)|$ for all $n \geq 9$, although we do not have a proof of this. It is interesting to note that, at least for $n \leq 150$, both the local maxima of $|\Delta_1(n)|$ and the local minima of $|\Delta_2(n)|$ occur precisely when $n \equiv 2 \pmod 3$. On the other hand, $|\Delta(n)|$ does not show such periodic extrema.

6. Triangular subpartitions and a combinatorial proof of Lipatov's formula for balanced words

Let $I(\tau) = |\{\zeta \in \Delta : \zeta \subseteq \tau\}|$ denote the number of triangular subpartitions of $\tau \in \Delta$. We start by giving a recurrence for this number. In the case where τ is a staircase, we obtain an explicit formula too, deriving a new proof of Theorem 4.1 in the process.

Let c^- and c^+ be the leftmost and rightmost removable cells of τ. Following the notation in [1], let τ° be the triangular partition obtained from τ by removing all the cells in the segment between c^- and c^+ (or, if $c^- = c^+$, just removing that cell).

Lemma 6.1. *For any $\tau \in \Delta(n)$ with $n \geq 1$,*

$$I(\tau) = I(\tau \setminus \{c^-\}) + I(\tau \setminus \{c^+\}) - I(\tau^\circ) + 1.$$

This recurrence relation comes from an inclusion-exclusion argument. Along with the base case $I(\epsilon) = 1$ (where ϵ denotes the empty partition), it allows us to compute $I(\tau)$ for any $\tau \in \Delta$, although not very efficiently. We will now present a more convenient formula for the case in which τ is a staircase.

We use the terms *height* and *width* of a partition τ to refer to the number of parts and the largest part of τ, respectively. Let $\Delta^{\ell\times\ell}$ be the set of triangular partitions whose width and height are at most ℓ. It can be proved that a partition belongs to $\Delta^{\ell\times\ell}$ if and only if it is a triangular subpartition of σ^ℓ. Our next goal is to give a formula for $I(\sigma^\ell) = |\Delta^{\ell\times\ell}|$. The proof of the following lemma uses the bijection ω from equation (2).

Lemma 6.2. *For $\ell \geq 1$, the number of triangular partitions of width exactly ℓ and height at most ℓ is $|\mathcal{B}_\ell|/2$, and*

$$|\Delta^{\ell\times\ell} \setminus \Delta^{(\ell-1)\times(\ell-1)}| = I(\sigma^\ell) - I(\sigma^{\ell-1}) = |\mathcal{B}_\ell| - 1.$$

Combining the above lemma with Lipatov's Theorem 4.1 enumerating balanced words, we deduce the following result.

Theorem 6.3. *For any $\ell \geq 0$,*

$$|\Delta^{\ell\times\ell}| = I(\sigma^\ell) = 1 + \sum_{i=1}^{\ell} \binom{\ell - i + 2}{2} \varphi(i).$$

Unfortunately, the proof of Theorem 6.3 using Lemma 6.2 and Lipatov's formula does not give a conceptual understanding of why the terms $\binom{\ell-i+2}{2}$ and $\varphi(i)$ appear.

Instead, we have been able to find a direct combinatorial proof of Theorem 6.3 that explains the role of these terms. Since the whole proof does not fit in this article, we will briefly outline its main ideas. First, we establish a bijection ϕ between triangular partitions that contain the cell $(2,1)$ and the set $\{(a,b,d,e) \in \mathbb{N}^4 \mid d < a, \gcd(d,e) = 1\}$, and characterize the image of $\Delta^{\ell\times\ell}$ by ϕ. Then, for a fixed pair of coprime numbers $d < e$, we take the union of the points (a,b) for which $(a,b,d,e) \in \phi(\Delta^{\ell\times\ell})$ and an affine transformation of the points (a,b) for which $(a,b,e,e-d) \in \phi(\Delta^{\ell\times\ell})$. The resulting set is formed by the lattice points inside a certain triangle, which are counted by $\binom{\ell-e+2}{2}$. Summing over all coprime pairs $d < e$ and taking into account some technical details, we obtain the formula in Theorem 6.3.

As an added benefit, our argument also provides a new proof of Lipatov's formula (Theorem 4.1).

7. Pyramidal partitions

In this section, we will study a higher-dimensional analogue of triangular partitions. These objects are first defined in [11], and some bounds on their growth are given in [13].

Definition 7.1. A d-dimensional pyramidal partition is a finite set of points in $\mathbb{N}^d$ that can be separated from its complement by a hyperplane.

Notice that a 2-dimensional pyramidal partition is the Ferrers diagram of a triangular partition. Proposition 3.1 can be extended to this more general setting; however, Lemma 2.6 does not hold anymore.

Theorem 7.2. *Let $d \in \mathbb{N}$. A finite nonempty subset $\pi \subset \mathbb{N}^d$ is a d-dimensional pyramidal partition if and only if* $\mathrm{Conv}(\pi) \cap \mathrm{Conv}(\mathbb{N}^d \setminus \pi) = \emptyset$.

Proposition 7.3. *For any $d \geq 3$, there are d-dimensional pyramidal partitions with an arbitrarily large number of removable and addable cells.*

In the case of triangular partitions in $\mathbb{N}^2$, we have that the only partitions $\tau \in \Delta$ such that $\tau = \tau'$ (that is, they are symmetrical with respect to the line $x = y$) are the staircase partitions. From this fact, we can deduce that $|\Delta(n)| \equiv 1 \pmod 2$ when $n = \binom{m}{2}$ for some integer $m \geq 2$, and $|\Delta(n)| \equiv 0 \pmod 2$ otherwise. This approach can be extended to d-dimensional pyramidal partitions by studying an action of the symmetric group on them. We will denote by $\Delta_{d\mathrm{D}}(n)$ the set of d-dimensional pyramidal partitions of size n, to avoid confusion with $\Delta_1(n)$ and $\Delta_2(n)$ defined in Section 5.

Theorem 7.4. *Let $n, d \in \mathbb{N}$, with d a prime number. If there exists an integer $m \geq d$ such that $n = \binom{m}{d}$, then $|\Delta_{d\mathrm{D}}(n)| \equiv 1 \pmod d$. Otherwise, $|\Delta_{d\mathrm{D}}(n)| \equiv 0 \pmod d$.*

8. Convex and concave partitions

Convex partitions are defined by Dean Hickerson in [10, A074658], where the number of convex partitions of size n is counted for $n \leq 55$. The concept of concave partitions is essential to some Schur positivity conjectures (see [2, Conjecture 7.1.1]). In this section, we will extend our research on triangular partitions to these more general families, starting with some characterizations.

Definition 8.1. A partition λ is said to be *convex* (resp. *concave*) if its Ferrers diagram consists of the points in $\mathbb{N}^2$ that lie on or below some convex (resp. concave) curve.

Proposition 8.2. *Given a partition λ, the following are equivalent:*

1. *λ is convex (resp. concave).*
2. *λ can be obtained as the intersection (resp. union) of a finite number of triangular partitions.*
3. $\mathrm{Conv}(\lambda) \cap (\mathbb{N}^2 \setminus \lambda) = \emptyset$ *(resp.* $\lambda \cap \mathrm{Conv}(\mathbb{N}^2 \setminus \lambda) = \emptyset$*).*
4. *There exists a convex (resp. concave) region $R \subset \mathbb{R}^2_{\geq 0}$ such that $\lambda = R \cap \mathbb{N}^2$.*

Using these new concepts, we can give a new characterization for triangular partitions.

Corollary 8.3. *A partition is triangular if and only if it is convex and concave.*

However, this characterization does not generalize to higher dimensions (see [12]).

Removable and addable cells in the convex and concave settings are defined in an analogous way to Definition 2.5.

Proposition 8.4. *A cell $c = (a, b)$ is removable from a convex partition η if and only if it is a vertex of* $\mathrm{Conv}(\eta)$ *and $(a+1, b), (a, b+1) \notin \eta$. Similarly, a cell c' is addable to a concave partition ν if and only if it is a vertex of* $\mathrm{Conv}(\mathbb{N}^2 \setminus \nu)$.

To close the article, we will study the asymptotic growth of the number of convex or concave partitions. We will use $\bigcap(n)$ (resp. $\bigcup(n)$) for the set of convex (resp. concave) partitions of size n.

Theorem 8.5. *There exists a constant b and a function $\delta(n) \sim \frac{3^{2/3}}{2} n^{2/3}$ such that*

$$\frac{2^{\sqrt[3]{n}}\sqrt{2\sqrt[3]{n}-2}}{4\sqrt[3]{n}+4} \leq \left|\bigcap(n)\right| \leq \exp(b\sqrt[3]{n}\log n), \qquad \frac{2^{\sqrt[3]{4(n-\delta(n))}}}{\sqrt{2+2\sqrt[3]{4(n-\delta(n))}}} \leq \left|\bigcup(n)\right|.$$

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