

# Cayley graphs and endomorphism monoids

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## Resum (CAT)

El tema principal del treball és l'estudi de diverses famílies especials de grafs simples com a grafs de Cayley de monoides i semigrups (és a dir, que es poden obtenir a partir de grafs de Cayley traient-ne els arcs múltiples, les direccions i els loops), tot seguint la línia d'algunes preguntes presents al paper *On monoid graphs* de K. Knauer i G. Puig i Surroca.

A nivell estructural, el projecte intenta arribar a aquestes preguntes de manera natural, progressant des de propietats bàsiques sobre grafs i semigrups a alguns petits resultats originals.

**Keywords:** *algebraic graph theory, monoid graphs, semigroup theory*



## Abstract

The main objective of this work was to study some families of simple graphs with special properties as *monoid* or *semigroup graphs*, that is, proving under which conditions they could be seen as the *underlying simple graph* of a directed graph built via the Cayley graph construction.

More specifically, we introduce and talk about the basic tools of the field of algebraic graph theory (we discuss the basic properties of the aforementioned Cayley graph construction, and a fairly young generalization of it by Yongwen Zhu as seen in [4]), introduce some recent interesting results in the literature by many authors, mostly by K. Knauer and coauthors (as in references [1], [2], [3]) and try to put together a comprehensive guide to try and understand the main difficulties and ideas used in one of the main lines of work in the field.

These motivations can be observed in our in-depth study of some families of outerplanar graphs as monoid graphs, which culminates in a small original characterization of outerplanar monoid graphs that admit a representation of the form  $\text{Cay}\{M, \{a, a^2\}\}$  for a monoid  $M$  and an element  $a \in M$ ; or our brief study of  $K_4 \sqcup C_5$  as a non monoid but possibly semigroup graph.

Both of the main lines of original work presented in the project were originally motivated by the work of K. Knauer and Puig i Surroca, as an attempt to answer part of the Questions 6.2 and 6.3 they posed in [3].

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## References

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