Extended Abstracts

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Counting subgroups using Stallings automata and generalisations

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Resum *(CAT)*
El problema de comptar els subgrups d'índex finit del grup lliure va ser abordat el 1949 per Marshall Hall, que va proporcionar una fórmula recursiva per al nombre de subgrups d'un índex finit donat en un grup lliure de rang finit. Aquest treball proporciona una prova del resultat de Hall utilitzant la teoria dels autòmats de Stallings. A més, veurem com obtenir una fórmula similar en el cas dels grups lliure per lliure-abelians, fent servir una generalització de la teoria dels autòmats de Stallings per a la família de grups lliure per lliure-abelians.

Keywords: *Stallings automata, enriched automata, finite index subgroup.*

Abstract

It is known that, in a finitely generated group, there are finitely many subgroups of a fixed finite index. It is therefore very natural to ask how many subgroups of a certain finite index there are in a group. In 1949, Marshall Hall Jr. answered this question in the case of free groups, providing a recursive formula for the number of subgroups of a given finite index in $\mathbb{F}_n$, the free group of rank n. In this work, we show how Hall's result can be understood and proved using the theory of Stallings automata. In addition, we present a recently developed generalisation of the theory of Stallings automata to the case of free times free-abelian groups and apply it to obtain a formula for the number of subgroups of a given finite index in $\mathbb{F}_n \times \mathbb{Z}^m$.

Since free times free-abelian groups are the direct product of a free group and a free-abelian group, we start by studying the problem of counting finite index subgroups in each of the factors separately.

In the free-abelian case, one can obtain a recursive formula for the number of subgroups of a given finite index in $\mathbb{Z}^m$ applying techniques similar to those of linear algebra. Essentially, a bijection is established between the subgroups of index k of $\mathbb{Z}^m$ and certain kind of matrices with integer coefficients and determinant equal to k. Using a recursive argument to count these matrices, the desired formula is obtained.

In the case of the free group, the corresponding formula was obtained by M. Hall in 1949 and it is possible to reformulate his proof using the celebrated theory of Stallings automata.

In 1983, Stallings established the basis for the study of the subgroups of the free group by means of a graphical representation consisting in a certain type of directed labelled graphs which are now known as Stallings automata. In this graphical representation, the elements of the subgroup correspond with the labels of certain closed walks in the Stallings automata. The key of this theory is the obtention of a bijection between subgroups of the free group and Stallings automata. The techniques of the theory of Stallings automata have allowed to solve in the free group many of the algorithmic problems which are usually posed in group theory (for example, the finite index problem, the intersection problem or the membership problem).

In particular, to arrive at Hall's formula, one can exhibit a map from the set of n-tuples of k permutations to the set of Stallings automata whose arcs are labelled with n elements and which are saturated and have k vertices (these objects are in bijection with the subgroups of $\mathbb{F}_n$ that have index k). Analysing the cardinal of the fibers of this map, one can deduce Hall's formula.

The classical theory of Stallings has been extended to subgroups of free times free-abelian groups. In this case, one can establish a bijection between subgroups of $\mathbb{F}_n \times \mathbb{Z}^m$ and certain type of automata enriched with abelian labels that encode the information corresponding to the abelian part of these groups. As an application of this bijection, we give a geometric characterisation of the subgroups of a given finite index in $\mathbb{F}_n \times \mathbb{Z}^m$ and, combining this characterisation with the existing formulae for the free and free-abelian cases, we obtain a new formula for the number of subgroups of a given finite index in $\mathbb{F}_n \times \mathbb{Z}^m$. For further reference see [1, 2, 3, 4].

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