Extended Abstracts

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Limit distribution of Hodge spectral exponents of plane curve singularities

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Resum *(CAT)*
K. Saito va formular la pregunta de si una distribució contínua és el límit de la distribució a l'interval $[0,1)$ dels exponents espectrals de Hodge d'una hipersuperfície, quan aquesta es mou en un sentit que s'ha de precisar. Ell ho va demostrar per a corbes planes irreductibles amb un límit molt específic. En aquest treball ens centrem en el cas de corbes planes irreductibles, per les quals explorem diferents formes d'assolir la distribució límit. També estudiem a on la distribució contínua és una cota superior per a la funció de distribució acumulada d'aquests invariants.

Keywords: *Hodge spectral exponents, plane curve singularities.*

Abstract

In the context of local Algebraic Geometry, we focus on the problem of understanding isolated singularities of complex hypersurfaces, described by an analytic equation $f \in \mathbb{C}\{x_0, \dots, x_n\}$ in a neighbourhood of the origin. In this work we study the distribution of a set of numerical invariants of f, the Hodge spectral exponents, for the particular case of irreducible plane curve singularities.

The Hodge spectral exponents $\alpha_1 \leqslant \cdots \leqslant \alpha_\mu$ are a set of μ rational numbers in the interval $(0, n+1) \subset \mathbb{R}$, where μ is the Milnor number and $\mathbb{C}^{n+1}$ is the ambient space of our hypersurface. This set is symmetric with respect to $\frac{n+1}{2}$ and is preserved by deformations (of the hypersurface) with constant Milnor number. The definition of the Hodge spectral exponents is based on the construction by Steenbrink of a mixed Hodge structure on the cohomology of the Milnor fiber. In addition, they are related to other major invariants of singularities such as the jumping numbers, which coincide with the Hodge spectral exponents in the interval $[0,1)$, and for plane curves both sets provide the same information on the singularity. Another related invariant is the geometric genus, which satisfies the relation $p_g = \#\{i | \alpha_i \leqslant 1\}$.

Kyoji Saito introduced the characteristic function $\chi_f(T)$ as the normalized spectrum, or equivalently as the Fourier transform of the (discrete) distribution of Hodge spectral exponents $D_f(s)$. By calculating the characteristic function he noticed that, for some sequences of hypersurfaces, the distribution of Hodge spectral exponents converges to a certain continuous distribution $N_{n+1}(s)$, which only depends on the dimension. Following this, K. Saito asked two main questions:

- For which sequences of hypersurfaces does the distribution of Hodge spectral exponents converge to $N_{n+1}(s)$?
- For which values $r \in \left(0, \frac{n+1}{2}\right) \subset \mathbb{R}$ is the cumulative distribution of $N_{n+1}(s)$ up to r an upper bound for the cumulative distribution of $D_f(s)$ up to r?

The main goal of this work is to provide partial answers to K. Saito's questions. We work within the case of irreducible plane curves, for which we have Morihiko Saito's explicit formula for the Hodge spectral exponents, and equivalently for the characteristic function. To have a better understanding of the problem, we elaborate on how the Fourier transform relates different concepts appearing in K. Saito's paper.

Regarding the convergence of the distribution of Hodge spectral exponents, K. Saito gave a partial result for irreducible plane curves, taking a very specific limit in terms of the last Puiseux pair of the curve. In this work we compute more general limits with respect to the Puiseux pairs. Consequently, we see that for some limits the distribution $D_f(s)$ does not converge to $N_2(s)$ but to other distributions.

With respect to the cumulative distribution, we prove a closed formula for $\phi_f(r)$, defined as the difference of the cumulative distributions for $N_2(s)$ and $D_f(s)$. This new tool allows us to give an alternative proof to a restricted version of a result by Tomari that states $\#\left\{i|\alpha_i \leqslant \frac{1}{2}\right\} < \frac{\mu}{8}$. In addition, it provides the means to bound $\phi_f(r)$ and thus determine intervals of values r where $\phi_f(r)$ is positive (or negative).

Moreover, thanks to this closed formula for $\phi_f(r)$, we are be able to calculate limits of the distribution of Hodge spectral exponents with a different procedure. This way we prove more general theorems on for which limits does the distribution $D_f(s)$ converge to $N_2(s)$, mainly in terms of the log-canonical threshold $\mathrm{lct}(f)$. For further reference see [1, 2, 3].

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