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Outerplanar partial cubes

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Resum *(CAT)*
Els partial cube-menors són una analogia de la noció de menors als partial cubes. En aquest article determinem el conjunt de pc-menors de les classes dels partial cubes outerplanars i els partial cubes sèrie-paral·lel. Aquest és el primer resultat d'aquest tipus per als partial cubes d'una classe tancada per menors.

Abstract *(ENG)*
Partial cube-minors are an analogue of graph minors in partial cubes. We determine the set of forbidden partial cube minors of the classes of outerplanar and series-parallel partial cubes. This is the first result of this type for the partial cubes in a minor closed graph class.

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1. Introduction

Denote by Q_d the hypercube graph of dimension d, i.e., its vertices are the elements of $\{0,1\}^d$ and two vertices are adjacent if they differ in exactly one entry. Partial cubes are the graphs that admit an isometric embedding into a hypercube; see Figure 1 for examples. They were introduced by Graham and Pollak [19] in the study of interconnection networks, form an important graph class in media theory [18], frequently appear in chemical graph theory [17, 20], and quoting [21], *present one of the central and most studied classes in Metric Graph Theory*. Some classes of partial cubes that are studied within Metric Graph Theory include median graphs [4], bipartite cellular graphs [3], hypercellular graphs [12], Pasch graphs [11], netlike partial cubes [24], and two-dimensional partial cubes [13]. Partial cubes arise also from geometry as graphs of regions of hyperplane arrangements in $\mathbb{R}^d$ [6], tope graphs of oriented matroids (OMs) [7], 1-skeleta of CAT(0) cube complexes [4], and more generally: tope graphs of complexes of oriented matroids [5].

An interesting structural feature of partial cubes is that they admit a natural minor-relation (pc-minors for short) consisting of restrictions and contractions, which are special forms of deletion and contraction in the graph. Many important classes of partial cubes are closed under taking pc-minors. Analogously to graph minors, given a pc-minor closed class there exists a list of excluded pc-minors of the class. Contrary to the situation of graph minors [25] for pc-minors this list might be infinite. If the list is finite, this also allows for a polynomial time recognition algorithm of the class [23]. Even if the list is infinite, determining it can yield insight into the class. All excluded minors are known for tope graphs of complexes of oriented matroids [23], two-dimensional partial cubes [13], median graphs, bipartite cellular graphs, hypercellular graphs, and Pasch graphs [12]. See [22, Chap. 7.5] for more related material. Since pc-minors are special graph minors, one source for pc-minor closed classes of partial cubes is the class of partial cubes in a minor-closed graph class. In the present paper we analyze the first non-trivial instance of such a class: partial cubes that are outerplanar partial cubes, i.e., they admit a crossing-free drawing in the plane such that all vertices lie on the outer face. We give a full description of its infinite list of excluded pc-minors (Theorem 4.21). Further, we obtain the list for series-parallel partial cubes (Theorem 4.22). Our proof uses the excluded minors for these classes [9] and we discuss in Section 5 possible extensions to other pc-minor closed classes. This short version omits some proofs, which can be found in [26].

2. Partial cubes

All graphs $G = (V, E)$ occurring in this paper are simple, connected, and finite. The *distance* $d(u, v) := d_G(u, v)$ between two vertices u and v is the length of a shortest (u, v)-path, and the *interval* $I(u, v)$ between u and v consists of all vertices on shortest (u, v)-paths: $I(u, v) := \{x \in V : d(u, x) + d(x, v) = d(u, v)\}$. If this causes no confusion, we will denote the distance function of G by d and not d_G. An induced subgraph of G is called *convex* if it includes the interval of G between any two of its vertices. An induced subgraph H of G is *isometric* if the distance between any pair of vertices in H is the same as that in G. In particular, convex subgraphs are isometric. A graph $G = (V, E)$ is *isometrically embeddable* into a graph $H = (W, F)$ if there exists a mapping $\varphi \to V \to W$ such that $d_H(\varphi(u), \varphi(v)) = d_G(u, v)$ for all vertices $u, v \in V$, i.e., $\varphi(G)$ is an isometric subgraph of H. A graph G is called a *partial cube* if it admits an isometric embedding into the hypercube Q_d. From now on, we will suppose that a partial cube $G = (V, E)$ is an isometric subgraph of the hypercube Q_d, i.e., we will identify G with its image under the isometric embedding and its vertices will often be denoted as elements of $\{0, 1\}^d$. The minimal d such that G embeds isometrically into Q_d is called the *(isometric) dimension* of G. The edges of G are partitioned into so-called Θ-*classes*, i.e., $e\Theta e'$ iff both edges correspond to a switch in the same coordinate of Q_d. Denote by $\mathcal{E} = \{E_i : i \in [d]\}$ the equivalence classes of Θ. Sometimes we will refer to Θ as a function $\Theta\colon E(G) \to \mathcal{E}$. The Θ-classes can be characterized intrinsically and do not depend on the embedding [16].

2.1 Partial cube minors

Let $G = (V, E)$ be an isometric subgraph of the hypercube Q_d. Given $f \in [d]$, an *elementary restriction* consists in taking one of the two connected components $\rho_{f^-}(G)$ and $\rho_{f^+}(G)$ of $G \setminus E_f$. These graphs are isometric subgraphs of the hypercube $Q([d] \setminus \{f\})$. Now applying twice the elementary restriction to two different coordinates f, g, independently of the order of f and g, we will obtain one of the four (possibly empty) subgraphs. Since the intersection of convex subsets is convex, each of these four subgraphs is convex in G and consequently induces an isometric subgraph of the hypercube $Q([d] \setminus \{f, g\})$. More generally, a *restriction* is a convex subgraph $\rho_A(G)$ of G, where $A \in \{+, -, 0\}^{[d]}$, obtained by iteratively applying $\rho_{e^{A_e}}$ for all $A_e \neq 0$. The following is well-known:

Lemma 2.1 ([1, 2]). *The set of restrictions of a partial cube G coincides with its set of convex subgraphs. In particular, the class of partial cubes is closed under taking restrictions.*

For $f \in [d]$, we say that the graph G/E_f obtained from G by contracting the edges of the equivalence class E_f is an (f-)*contraction* of G. For a vertex v of G, we will denote by $\pi_f(v)$ the image of v under the f-contraction in G/E_f, i.e., if uv is an edge of E_f, then $\pi_f(u) = \pi_f(v)$, otherwise $\pi_f(u) \neq \pi_f(v)$. We will apply π_f to subsets $S \subset V$, by setting $\pi_f(S) := \{\pi_f(v) : v \in S\}$. In particular, we denote the f-*contraction* of G by $\pi_f(G)$. It is well-known and follows from the proof of the first part of [10, Thm. 3] that $\pi_f(G)$ is an isometric subgraph of $Q([d] \setminus \{f\})$. Since edge contractions in graphs commute, i.e., the resulting graph does not depend on the order in which a set of edges is contracted, we have:

Lemma 2.2. *Contractions commute in partial cubes, i.e., if $f, g \in [d]$ and $f \neq g$, then $\pi_g(\pi_f(G)) = \pi_f(\pi_g(G))$. Moreover, the class of partial cubes is closed under contractions.*

Consequently, for a set $A \subset [d]$, we can denote by $\pi_A(G)$ the isometric subgraph of $Q([d] \setminus A)$ obtained from G by contracting the classes $A \subset [d]$ in G. Finally, we have:

Lemma 2.3 ([12]). *Contractions and restrictions commute in partial cubes, i.e., if $f, g \in [d]$ and $f \neq g$, then $\rho_{g^+}(\pi_f(G)) = \pi_f(\rho_{g^+}(G))$.*

The previous lemmas show that any set of restrictions and any set of contractions of a partial cube G provide the same result, independently of the order in which we perform the restrictions and contractions. The resulting graph G' is also a partial cube, and G' is called a *partial cube-minor* (or *pc-minor*) of G.

2.2 Expansions and Cartesian products

A partial cube G is an *expansion* of a partial cube G' if $G' = \pi_f(G)$ for some equivalence class f of $\mathcal{E}(G)$. More generally, let G' be a graph containing two isometric subgraphs G'_1 and G'_2 such that $G' = G'_1 \cup G'_2$, there are no edges from $G'_1 \setminus G'_2$ to $G'_2 \setminus G'_1$, and $G'_0 := G'_1 \cap G'_2$ is nonempty. A graph G is an *isometric expansion* of G' with respect to G'_0 if G is obtained from G' by replacing each vertex v of G'_1 by a vertex v_1 and each vertex v of G'_2 by a vertex v_2 such that u_i and v_i, $i = 1, 2$, are adjacent in G if and only if u

and v are adjacent vertices of G'_i and v_1v_2 is an edge of G if and only if v is a vertex of G'_0. Every partial cube can be obtained from a single vertex by a sequence of expansions [10].

The *Cartesian product* $F_1 \square F_2$ of two graphs $F_1 = (V_1, E_1)$ and $F_2 = (V_2, E_2)$ is the graph defined on $V_1 \times V_2$ with an edge $(u, u')(v, v')$ if and only if $u = v$ and $u'v' \in E_2$ or $u' = v'$ and $uv \in E_1$. Cartesian products of partial cubes are partial cubes. It follows immediately from the definitions that:

Lemma 2.4. *A partial cube G is an expansion of the partial cube G' if and only if $G' \subseteq G \subseteq G' \square K_2$ are isometric subgraphs.*

3. The excluded minors

A graph is *outerplanar* if it admits a planar drawing for which all vertices lie on the outer face of the drawing. This class is minor-closed hence also outerplanar partial cubes have a set of excluded pc-minors, which we will denote by Ω. Denote by $L := K_{1,3} \square K_2$ the *book graph* and by $n \geq 3$, G_n is the *gear graph*, i.e., the graph formed by $2n + 1$ vertices: an even exterior cycle of length $2n$ and a center vertex adjacent to one bipartition class of the cycle.

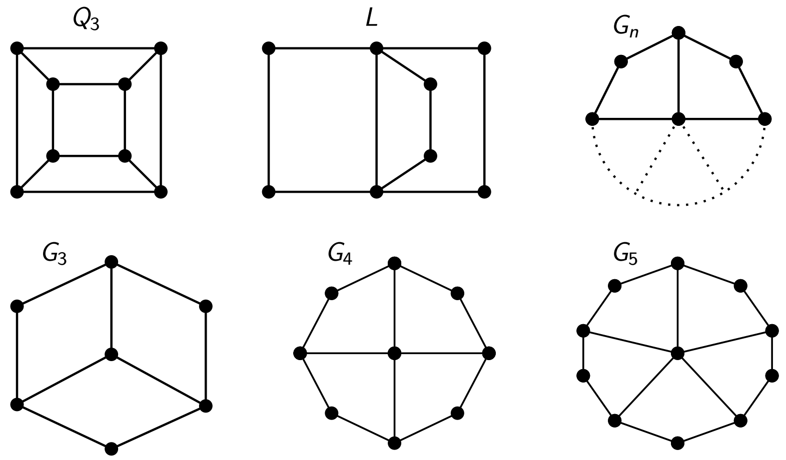

Figure 1: The cube, the book graph, and the infinite family of gear graphs.

It is easy to see that all the partial cubes in Figure 1 are pc-minor minimal non-outerplanar. Our main result is that they are the only such graphs. The proof will occupy the rest of this paper.

4. Main proof

4.1 Preparation

Before we get into the proof, we need some lemmas whose proofs are omitted in this short version.

Lemma 4.1. *If $G \in \Omega$, then G is planar.*

Let G be a graph, let F be a set of edges, let $H \subseteq G$ be a subdivision of a certain graph K. We say that F *destroys* H if H/F is not a subdivision of K. We say that F *destroys* K if G/F does not contain any subdivision of K as a subgraph.

Lemma 4.2. *Let $G \in \Omega$, let E_i be a Θ-class. Then E_i destroys K_4 or $K_{2,3}$. In particular, if $H \subseteq G$ is a subdivision of K_4 or $K_{2,3}$, then E_i destroys H.*

If $H \subseteq G$ is a subgraph, we refer to the induced subgraph by $V(H)$ as the induced subgraph by H and denote it as $G[H]$.

Lemma 4.3. *Let G be a graph. Let $H \subseteq G$ be a subdivision of a certain graph K. Let F be a matching. Then $F \backslash E(G[H])$ does not destroy H.*

If $H \subseteq G$ is a subgraph and F is a set of edges of G, then we denote $F[H] := F \cap E(G[H])$.

Lemma 4.4. *Let $G \in \Omega$. Let $H \subseteq G$ be a subdivision of K_4 or $K_{2,3}$. Let E_i be a Θ-class. Then $E_i[H] \neq \emptyset$.*

4.2 Three lemmas

Lemma 4.5. *Let G be a partial cube containing a subdivision of $K_{2,3}$ or K_4 such that no pc-minor of G does. If* $\dim(G) \leq 3$, *then $G = G_3$ or $G = Q_3$.*

Proof. Partial cubes of dimension 0, 1, and 2 are all outerplanar. For dimension 3, note that any pc-minor of G will be a subgraph of Q_2, thus outerplanar. Among all partial cubes of dimension 3, the only ones containing a subdivision of $K_{2,3}$ or K_4 are G_3 and Q_3. □

From now we can restrict to partial cubes of isometric dimension at least 4. We start with those containing only a subdivision of $K_{2,3}$.

Lemma 4.6. *Let G be a partial cube with* $\dim(G) \geq 4$ *containing a subdivision of $K_{2,3}$ but none of K_4 such that no pc-minor of G contains a subdivision of $K_{2,3}$. Then $G = L$.*

Proof. Among all subdivisions of $K_{2,3}$ contained in G, we choose a $K_{2,3}$-subdivision H contained in G with the minimum number of vertices. Let a, b, c, d, z be the *original vertices* of $K_{2,3}$, with $\deg_H(a) = 3 = \deg_H(z)$. H consists in three disjoint paths $\overline{abz}$, $\overline{acz}$, and $\overline{adz}$ called *main paths*. Each one of these paths contains at least two edges in two different Θ-classes. We can assume that b, c, d are the first vertex in each main path respectively, i.e., $ab, ac, ad \in E(H)$. Let E_1, E_2, E_3 be Θ-classes such that $ab \in E_1$, $ac \in E_2$, $ad \in E_3$.

Claim 4.7. Let P be a main path. Let $u, v \in P$ such that $\{u, v\} \neq \{a, z\}$. If $uv \notin E(H)$, then $uv \notin E(G)$.

Proof. Assume $uv \notin E(H)$ and $uv \in E(G)$. Since $u, v \in P$, there is a vertex $w \in P$ between u and v such that $w \notin Q := \overline{auvz}$. Since $\{u, v\} \neq \{a, z\}$, $\ell(Q) \geq 2$. Also, $w \notin Q$ implies that $\ell(Q) < \ell(P)$. Let H' be the graph built from H and replacing P for Q. H' is a subdivision of $K_{2,3}$ with less vertices than H, which is a contradiction to the fact that H is minimal in vertices (Figure 2). □

We conclude that there are no induced edges between vertices contained in the same main path, except maybe between a and z.

Claim 4.8. Let $u, v \in H$, vertices from two different main paths. Any path in G between u and v goes through a or z. In particular, $uv \notin E(G)$.

Proof. Let P, Q be two main paths such that $u \in P$, $v \in Q$. Let R be a path between u and v such that $a, z \notin R$. Note that $u \in P \backslash \{a, z\}$, $v \in Q \backslash \{a, z\}$ are two disjoint paths. Assume that $P \cap R = \{u\}$ and $Q \cap R = \{v\}$. Now a K_4-subdivision is formed, picking as original vertices a, u, v, z and six main paths, where R is one of them and the others paths are contained in H (Figure 2). □

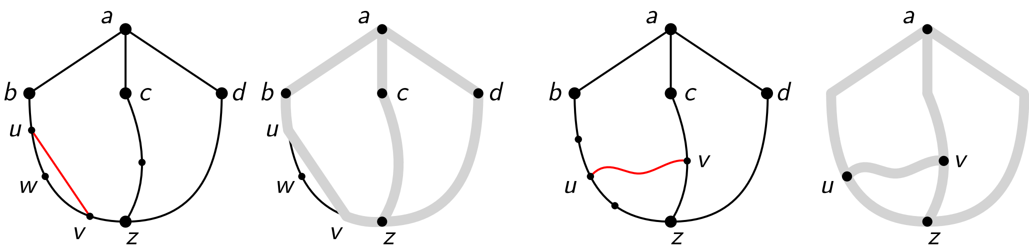

Figure 2: Claims 4.7 and 4.8: If uv exists, then: (left) there is a $K_{2,3}$-subdivision not containing w or (right) there is a K_4-subdivision.

Claims 4.7 and 4.8 imply that az will be (if it exists) the only edge in G induced by H.

Claim 4.9. a and z differ in only one coordinate, i.e., $az \in E(G)$.

Proof. Assume a and z differ in at least two coordinates, i.e., $a = (0, 0, \dots)$ and $z = (1, 1, \dots)$. Let E_1, E_2 be the Θ-classes corresponding to the first two coordinates. Since the three main paths are disjoint, there exist $e_{1b}, e_{1c}, e_{1d} \in E_1$ and $e_{2b}, e_{2c}, e_{2d} \in E_2$ such that $e_{1b}, e_{2b} \in \overline{abz}$, $e_{1c}, e_{2c} \in \overline{acz}$, $e_{1d}, e_{2d} \in \overline{adz}$. Then there exist three vertices $u_b \in \overline{abz}$, $u_c \in \overline{acz}$, $u_d \in \overline{adz}$ such that u_i is between e_{1i} and e_{2i} in each main path (Figure 3). Then each u_i has its first two coordinates either $(0, 1)$ or $(1, 0)$. In each eight combinations, at least two vertices have the same two first coordinates. Assume $u_b = (0, 1, \dots)$, $u_c = (0, 1, \dots)$. Now, let P be a short (u_b, u_c)-path. Any vertex $v \in P$ has got to have the same first two coordinates, i.e., $v = (0, 1, \dots)$. Then, neither a nor z can be in P. This is a contradiction with Claim 4.8. Then, a and z differ in only one coordinate, i.e., $az \in E(G)$. We can assume from now on that $az \in E_4$. □

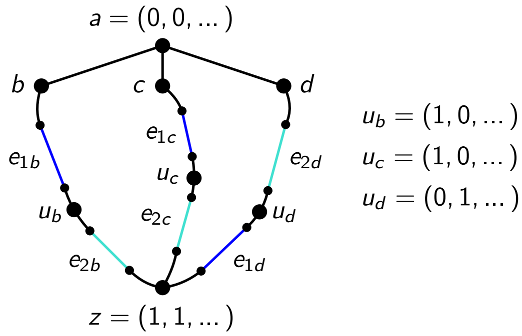

Figure 3: Claim 4.9: a short (u_b, u_c)-path cannot pass through a nor z.

Claim 4.10. Let P be a main path. Then, $\ell(P) = 3$ and $\Theta(P) = (E_i, E_4, E_i)$, where E_i is the Θ-class corresponding to the first edge of P starting from a, i.e., $i \in [3]$.

Proof. $P \cup \{az\}$ forms a cycle of length 4 or greater. Thus, this cycle has at least two edges in E_i and E_4. The other main paths Q, R already have three edges not contained in E_i. Then, $\pi_i(Q)$ and $\pi_i(R)$ do still have length greater than 2. Lemma 4.2 ensures that each Θ-class destroys H. Then, since E_i destroys H, we get $\ell(\pi_i(P)) < 2$. Thus, $\ell(\pi_i(P)) = 1$ and $\Theta(P) = (E_i, E_4, E_i)$. □

From Claim 4.10 we get to fully determine H. It turns out that $G[H] = H \cup \{az\} = L$.

Claim 4.11. $\dim(G) = 4$.

Proof. Thanks to Lemma 4.4, all Θ-classes have to contain an edge in $G[H]$, but $G[H] = L \subseteq Q_4$. □

Still, we have not fully determined $V(G)$ and there could be a vertex $v \in V(G) \backslash V(H)$.

Claim 4.12. $V(H) = V(G)$.

Proof. G is a partial cube, then G is connected. If $V(G) \backslash V(H) \neq \emptyset$, then there is a vertex $u \in V(G) \backslash V(H)$, adjacent to some $v \in V(H) \backslash \{a, z\}$. Assume $v = b$. Then either $bu \in E_2$ or $bu \in E_3$. Assume the first option. G is a partial cube implies $cu \in E(G)$ and $\Theta(cu) = E_1$. But that is a contradiction with Claim 4.8. Then $V(G) = V(H)$. □

Finally, $V(G) = V(H)$ and $G[H] = L$ imply that $G = L$, which finishes the proof of Lemma 4.6. □

Lemma 4.13. *Let G be a partial cube with* $\dim(G) = n \geq 4$ *containing a subdivision of K_4 such that no pc-minor of G contains a subdivision of K_f. Then $G = G_n$.*

Proof. Among all subdivisions of K_4 in G, we choose a subdivision H with the minimum number of vertices. Let a, b, c, d be the *original vertices* of K_4. The six edges of K_4 are called *main paths* in H. Let $e \in E(G[H])$. Then up to symmetry e has to be one of the following types (Figure 4):

(i) $e_1 = u_1v_1 \in E(H)$, $u_1, v_1 \in \{a, b, c, d\}$ are original vertices.

(ii) $e_2 = u_2v_2 \in E(H)$, $u_2 \in \{a, b, c, d\}$ is an original vertex and v_2 is a subdivision vertex of a main path containing u_2.

(iii) $e_3 = u_3v_3 \in E(H)$, u_3, v_3 are two subdivision vertices in the same main path.

(iv) $e_4 = u_4v_4 \notin E(H)$, $u_4 \in \{a, b, c, d\}$ is an original vertex and v_4 is a subdivision vertex of a main path that does not contain u_4.

(v) $e_5 = u_5v_5 \notin E(H)$, $u_5, v_5 \in \{a, b, c, d\}$ are original vertices.

(vi) $e_6 = u_6v_6 \notin E(H)$, $u_6 \in \{a, b, c, d\}$ is an original vertex and v_6 is a subdivision vertex of a main path containing u_6.

(vii) $e_7 = u_7v_7 \notin E(H)$, u_7, v_7 are two subdivision vertices of the same main path.

(viii) $e_8 = u_8v_8 \notin E(H)$, u_8, v_8 are two subdivision vertices of two adjacent main paths.

(ix) $e_9 = u_9v_9 \notin E(H)$, u_9, v_9 are two subdivision vertices of two opposite main paths.

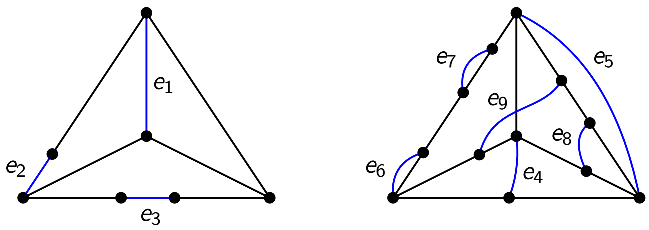

Figure 4: The nine different types of induced edges by H. On the left, the edges contained in H, on the right, the edges not contained in H.

Claim 4.14. Types (v), (vi), (vii), (viii), (ix) edges cannot exist (Figure 5).

Proof. **(v)** Assume $e_5 = ab \notin E(H)$. The main path $\overline{ab}$ cannot be a single edge. Thus, $\ell(\overline{ab}) \geq 2$. Then, there exists a vertex $w \in \overline{ab}$, $u \neq a, b$. Then, a K_4-subdivision H' is formed with the same original vertices a, b, c, d and the same main paths but replacing $\overline{awb}$ for the edge $e_5 = ab$. H' contains less vertices than H, contradiction.

(vi) Assume $e_6 = au \notin E(H)$. a and u are not adjacent in H. There is a vertex $w \in \overline{ab}$ between a and u. Then, there is a subdivision H' of K_4 with the same original vertices a, b, c, d and the same main paths but replacing $\overline{awub}$ for the path $\{au\} \cup \overline{ub}$. H' contains less vertices than H, contradiction.

(vii) Assume $e_7 = uv \notin E(H)$, $u, v \in \overline{ab}$. There exists a vertex $w \in \overline{ab}$ between u and v. Then there is another subdivision H' with the same original vertices a, b, c, d and the same main paths but replacing $\overline{auwvb}$ for the path $\overline{au} \cup \{uv\} \cup \overline{vb}$. H' contains less vertices than H, contradiction.

(viii) Assume $e_8 = uv \notin E(H)$, $u \in \overline{ab}$ and $v \in \overline{ac}$ are two subdivision vertices. There is a cycle going through a, u, v and at least a fourth vertex $w \in H$ (due to G being a partial cube). Assume $w \in \overline{au} \subset \overline{ab}$. Then there is another subdivision H' with original vertices v, b, c, d and the three main paths containing v being: $\overline{vd}, \overline{va} \cup \overline{ac}, \overline{bu} \cup \{uv\}$. H' contains less vertices than H, contradiction.

(ix) Even though we can find a subdivision of K_4 that has less vertices than H, there is another argument we can do. Assume $e_9 = uv$, $u \in \overline{ab}$, $v \in \overline{cd}$. Then, $H \cup \{uv\} = K_{3,3}$, where the bipartition is $V(K_{3,3}) = \{a, b, v\} \cup \{c, d, u\}$. That means G is not planar, which is a contradiction to Lemma 4.1. □

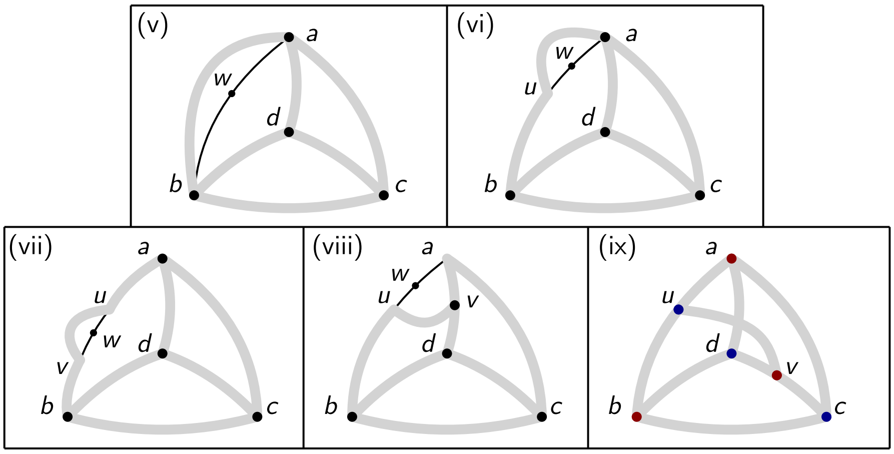

Figure 5: Representation of cases (v), (vi), (vii), (viii), (ix). In grey, the subdivisions of K_4 or $K_{3,3}$ deduced from the hypothesis of each case.

Now we have that $G[H] \backslash H$ can only have edges of type (iv), which are called *mixed edges*. Edges of type (i) are called *original edges* and edges of types (ii) and (iii) are called *subdivision edges*.

Claim 4.15. Let E_i be a Θ-class. E_i contains an original edge o mixed edge (types (i) o (iv)).

Proof. Thanks to Lemma 4.4, we know $E_i[H] := E_i \cap E(G[H]) \neq \emptyset$, since $G \in \Omega$ and H is a subdivision of K_4. Assume every edge in $E_i[H]$ is type (ii) or (iii), i.e., they are all subdivision edges. Contract all edges of $E_i \backslash E_i[H]$ (edges in E_i not induced by H). Lemma 4.3 implies $H = H/(E_i \backslash E_i[H])$, i.e., H is not affected by the contraction of $E_i \backslash E_i[H]$. Now, if we contract $E_i[H]$, we will have contracted all edges of E_i. Due to Lemma 4.2, $\pi_i(G)$ will not contain any subdivision of K_4. However, we are assuming all edges in $E_i[H]$ are subdivision edges, i.e., all edges in $E_i[H]$ are contained inside the main paths. There cannot be any main path containing only edges in E_i (except if the main path is a single edge, but in that case it would be an original edge). Then $\pi_i(H)$ still contains the same main paths contracted, but never until being fully contracted. Then, $\pi_i(G)$ contains $\pi_i(H)$ as a subgraph, which is still a subdivision of K_4. That is a contradiction which means that E_i has to have an original edge or a mixed edge (types (i) and (iv)). □

Claim 4.16. G contains at least one mixed edge (type (iv)).

Proof. We proof G cannot have more than three original edges. Since $n := \dim(G) \geq 4$, there is at least one Θ-class containing mixed edge. Assume E_1, E_2, E_3 are Θ-classes each one containing an original edge. Except symmetries, they can only form a C_3, P_3 or $K_{1,3}$ inside K_4. Let E_4 be a Θ-class. A fourth original edge in E_4 would form a C_4 or a $C_3 +_1 P_1$ together with the other three. A C_4 in a partial cube cannot have four different Θ-classes and a $C_3 +_1 P_1$ has an odd cycle, thus, E_4 cannot contain an original edge. Then, Claim 4.15 implies that E_4 necessarily contains a mixed edge. Moreover, G contains at least $n-3$ mixed edges. □

Claim 4.17. All mixed edges are incident to the same original vertex.

Proof. Let $e, f \in E(G)$ be two mixed edges incident to two different originals vertices. Assume $e = au$ and $f = dv$, $u, v \in V(H)$, being two subdivision vertices. Up to symmetries we have four cases; see Figure 6:

(i) $u, v \in \overline{bc}$.
(ii) $u \in \overline{bd}$ and $v \in \overline{bc}$.
(iii) $u \in \overline{bd}$ and $v \in \overline{ac}$.
(iv) $u \in \overline{bd}$ and $v \in \overline{ab}$.

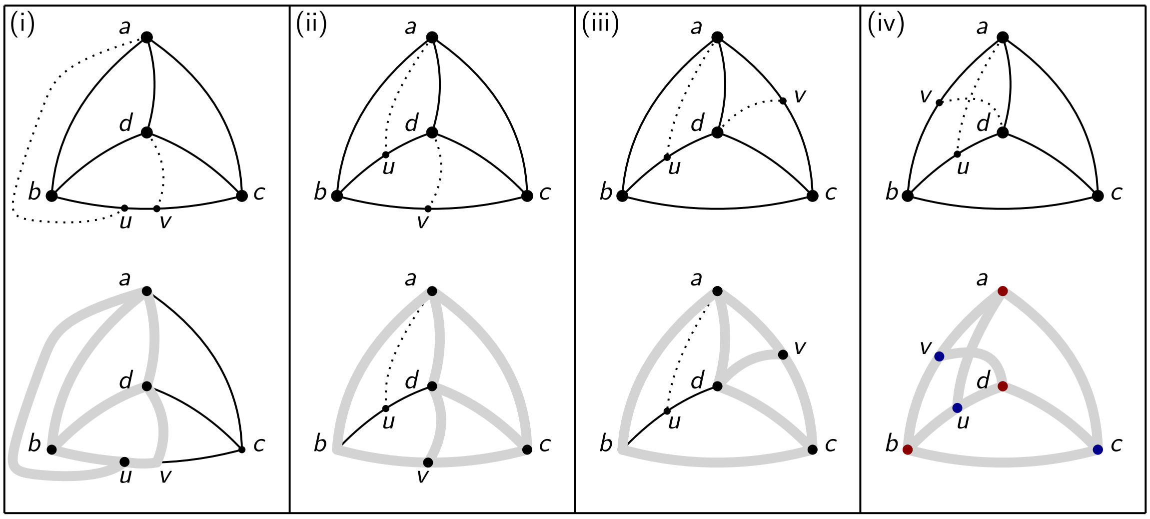

Figure 6: Cases (i), (ii), (iii), and (iv) of Claim 4.17.

In cases (i), (ii), (iii), just like in Figure 5 we can find a subdivision of K_4 with strictly less vertices than H. In case (iv) we can find a subdivision of $K_{3,3}$, contradicting that it is planar by Lemma 4.1. Hence, all mixed edges are incident to the same original vertex. □

We can assume all mixed edges are incident to d.

Claim 4.18. The main paths $\overline{ad}$, $\overline{bd}$, $\overline{cd}$ are indeed original edges, i.e., $ad, bd, cd \in E(H)$.

Proof. Claim 4.16 says there is at least a mixed edge $e \in E(G[H])$. Assume $e = du$, $u \in \overline{bc}$. There are three K_4-subdivisions H_1, H_2, H_3 taking as original vertices $\{b, c, d, u\}$, $\{a, c, d, u\}$, $\{a, b, d, u\}$, respectively. H having the minimum number of vertices implies $ad, bd, cd \in E(H)$. □

Now we know H contains three original edges and $n-3$ mixed edges. Thus, $\deg_G(d) = n$. We still need to know about the outer cycle of H, $Z := \overline{abca}$. From now on, we will not differentiate between the original vertices a, b, c and the other vertices in Z adjacent to d through a mixed edge. We will denote as $v_1, \dots, v_n \in Z$ the vertices adjacent to d in G, ordered consecutively, and $E_1, \dots, E_n$ the Θ-classes of edges $dv_1, \dots, dv_n$, respectively. Analogously, we will not differentiate H from any other subdivision of K_4 taking d and any three vertices $v_i \in Z$, since they all have the same number of vertices (minimal, by hypothesis). $\forall i$, let $P_i := \overline{v_i v_{i+1}} \subseteq Z$ be the path not containing any other v_j (Figure 7(i)).

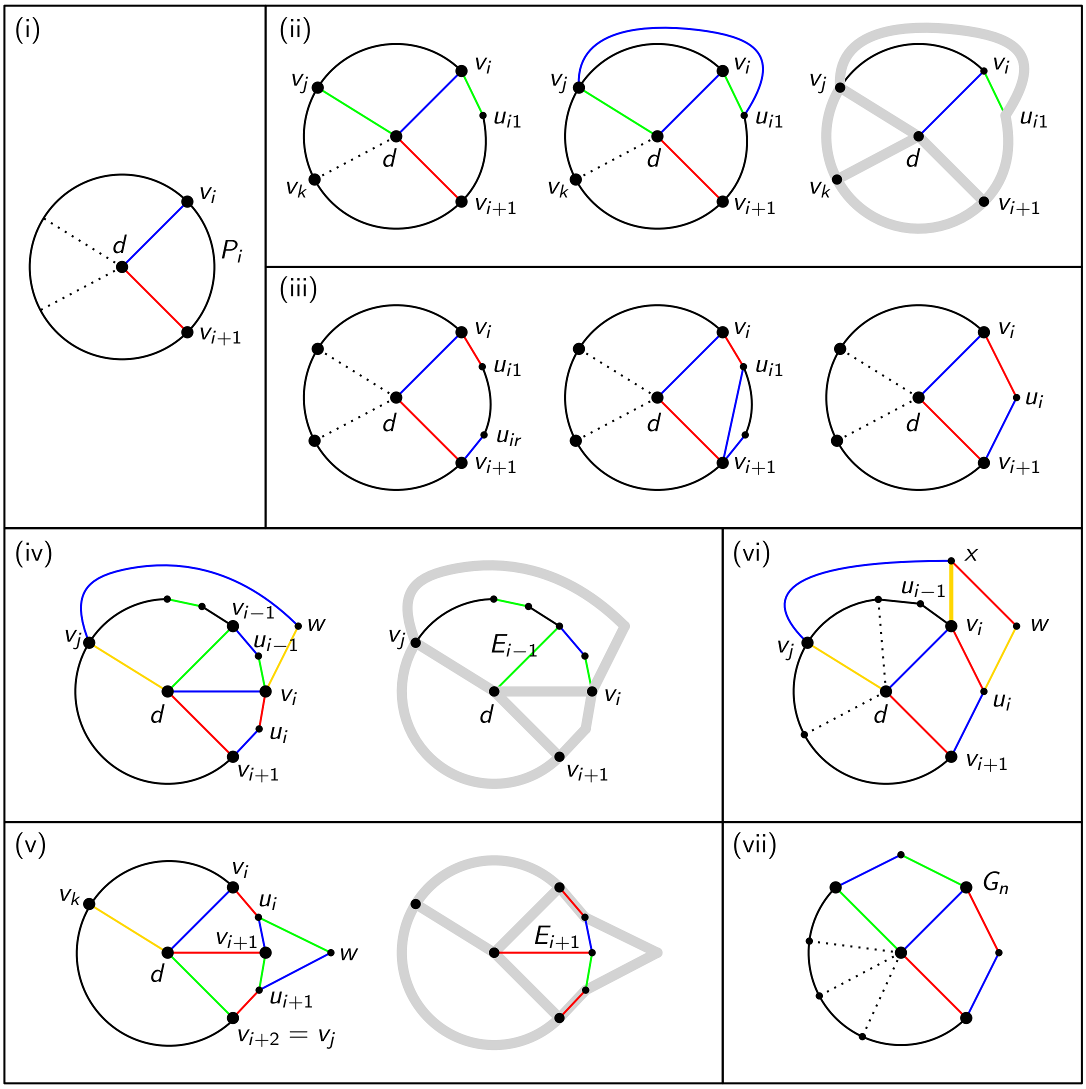

Figure 7: Summary of the different steps and cases in Claims 4.19 and 4.20.

Claim 4.19. $\text{Long}(P_i) = 2$ and $\Theta(P_i) = (E_{i+1}, E_i)$.

Proof. Assume $P_i = (v_i, u_{i1}, \dots, u_{ir}, v_{i+1})$. Keeping in mind that $n \geq 4$ and H is minimal in vertices, we can deduce that $v_i u_1 \in E_{i+1}$ and $u_r v_{i+1} \in E_i$ (Figure 7(ii)). Using partial cube properties, we deduce that $u_1 = u_r$. Thus, $\ell(P_i) = 2$ and $\Theta(P_i) = (E_{i+1}, E_i)$ (Figure 7(iii)). □

We deduce from Claim 4.19 that $Z = (v_1, u_1, v_2, u_2, \dots, v_n, u_n, v_1)$ and $G_n \subseteq G$. Moreover, we have that $G_n = G[H] = H \cup \{dv_i, 1 \leq i \leq n\}$.

Claim 4.20. $V(G) = V(G_n)$.

Proof. Assume $w \in V(G) \backslash V(H)$ is adjacent to a vertex $v \in V(H)$. v has to be in Z. We have two options:

(i) $v = v_i \in Z$, $1 \leq i \leq n$.

(ii) $v = u_i \in Z$, $1 \leq i \leq n$.

(i) Assume $wv_i \in E(G)$ and $wv_i \in E_j$. Note that $j \neq i-1, i, i+1$, and can exist since $n \geq 4$ (if $n = 4$, then there is only one option for j). To complete a square, we must have $wv_j \in E(G)$, $wv_j \in E_i$. Then there is a new K_4-subdivision with original vertices d, v_i, v_{i+1}, v_j, in which E_{i-1} does not contain any induced edge by H'. But this cannot happen, as Lemma 4.4 affirms that $E_{i-1}[H'] \neq \emptyset$ (Figure 7(iv)).

(ii) Assume $wu_i \in E(G)$, $wu_i \in E_j$. Note that $j \neq i, i+1$. We split it in two new cases for j:

(a) $j = i-1$ or $j = i+2$, i.e., v_j is consecutive to v_i or v_{i+1} in Z.

(b) $j \neq i-1, i+2$, i.e., v_j is not consecutive to v_i nor v_{i+1} in Z (cannot happen if $n = 4$).

(a) By symmetry, assume $j = i+2$. The square $\{w, u_i, v_{i+1}, u_{i+1}\}$ is completed, so we have $wu_{i+1} \in E_i$. But note that now G contains a K_4-subdivision H' with original vertices d, v_i, v_{i+2}, v_k, where k can exist since $n \geq 4$ (Figure 7(v)). Note that $|H'| = |H|$ and E_{i+1} does not contain any original or mixed edge in H', which contradicts Claim 4.15.

(b) Assume $j \neq i-1, i+2$. The path $(w, u_i, v_{i+1}, d, v_j)$ has length 4 and two edges in E_j. Then a short path $P = \overline{wv_j}$ has to have length 2, i.e., $P = (w, x, v_j)$, $x \notin Z$, and $\Theta(P) = (E_{i+1}, E_i)$. This forces the edge xv_i to exist and be contained in E_j. $v_1 x$ satisfies case (i) conditions, which we have already seen that it leads to a contradiction (see Figure 7(vi)).

Every case leads to absurdity. Then, there are no edges vw between $v \in Z$ and $w \in V(G) \backslash V(G_n)$. Since G is connected, we get $V(G) \backslash V(G_n) = \emptyset$, i.e., $V(G) = V(G_n)$. □

We have that $G[G_n] = G$ and $V(G_n) = V(G)$. Finally we conclude that $G = G_n$ (Figure 7(vii)), which finishes the proof of Lemma 4.13. □

4.3 Final results

Theorem 4.21. *The excluded pc-minors for outerplanar partial cubes are L, Q_3, and G_n for $n \geq 3$.*

Proof. Let G be a non-outerplanar partial cube such that every pc-minor of G is outerplanar. Chartrand–Harary [9] prove that non-outerplanar graphs contain $K_{2,3}$ or K_4 as a minor and it is easy to see that hence they contain a subdivision of $K_{2,3}$ or K_4. In particular this holds for G. By Lemmas 4.5, 4.6, and 4.13 we obtain that any pc-minor minimal non-outerplanar partial cubes must be a member of $\{L, Q_3, G_n, n \geq 3\}$. The proof that all elements of $\{L, Q_3, G_n, n \geq 3\}$ pc-minor minimal non-outerplanar partial cubes can be found in [26]. □

Since Lemmas 4.5, 4.6, and 4.13 are very specific concerning the graph that is obtained as a subdivision and series-parallel graphs are exactly those not containing a subdivision of K_4, see e.g. [8], we get:

Theorem 4.22. *The excluded pc-minors for series-parallel partial cubes are Q_3 and G_n for $n \geq 3$.*

5. Conclusions

The next natural minor-closed class are planar partial cubes, which have been characterized in different ways [1, 14]. Computer experiments show that in isometric dimensions 4, 5, 6 there are already $9 + 61 + 272 = 344$ pc-minor-minimal non-planar partial cubes. Considering pc-minor-minimal non-planar partial cubes such that all their isometric subgraphs are planar yields $2 + 10 + 34 = 46$ graphs. Looking only at pc-minor-minimal non-planar median graphs gives $1 + 4 + 8 = 13$ obstructions. Another possible class to attack are apex-outerplanar partial cubes, i.e., graphs that become outerplanar after removing some vertex. This minor-closed class lies between outerplanar and planar graphs, its 57 excluded minors are known; see [15]. For any excluded pc-minor G of outerplanar partial cubes, $G \square K_2$ is an excluded pc-minor of apex-outerplanar partial cubes as well as for planar partial cubes, i.e., in both cases the list is infinite.

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