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Weak convergence of the Lazy Random Walk to the Brownian motion

*Salim Boukfal Lazaar
Universitat Autònoma de Barcelona (UAB)
salim.boukfal.lazaar@gmail.com

∗Corresponding author

Resum *(CAT)*
En aquest article considerem una modificació del passeig aleatori simple, el *Lazy Random Walk*, i construïm una família de processos estocàstics a partir d'aquest procés que convergeix feblement cap a un moviment Brownià estàndard en una dimensió.

Abstract *(ENG)*
In this paper we consider a modification of the simple random walk, the Lazy Random Walk, and construct a family of stochastic processes from the latter that converges weakly to a standard one-dimensional Brownian motion.

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1. Introduction

Brownian motion can be thought, intuitively, as the random motion of particles suspended in a medium (usually liquid or gas) or as well as a stochastic process with “small” and independent displacements which are independent of the position of the particle. A formal definition containing all these features can be given as follows:

Definition 1.1. A stochastic process $\{B_t : t \geq 0\}$ is a standard one-dimensional Brownian motion if:

(i) $B_0 = 0$ almost surely.

(ii) For any $k \in \mathbb{N}$ and any $0 \leq t_1 < \cdots < t_k < \infty$, the random variables $B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_k} - B_{t_{k-1}}$ are independent.

(iii) For any $0 \leq s < t < \infty$, the random variable $B_t - B_s$ is normally distributed with zero mean and $t - s$ variance.

(iv) The sample paths of the process are continuous everywhere.

However, this last definition includes a couple of properties that do not arise trivially from the intuitive point of view. For instance, why should be the displacements normally distributed or why should the sample paths be continuous functions of the time variable? Besides, there is no guarantee that there is such a mathematical object satisfying all those properties at the same time.

Donsker's Invariance Principle allows us to connect the intuition with mathematical formalism and states that, whenever the displacements are small enough (they have finite variance), we can construct a family of stochastic processes converging weakly (or in law) to a stochastic process whose law verifies Definition 1.1. More particularly:

Theorem 1.2 (Donsker's Invariance Principle)**.** *Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of independent and identically distributed random variables with mean $\mu \in \mathbb{R}$ and variance $0 < \sigma^2 < \infty$ and let $\xi_j = X_j - \mu$. Then the random (continuous) functions*

$$Y_t^{(n)} = \frac{1}{\sigma\sqrt{n}} S_{nt}, \quad 0 \leq t \leq 1, \tag{1}$$

where

$$S_t = \sum_{j=1}^{[t]} \xi_j + (t - [t])\xi_{[t]+1}, \quad S_0 = 0,$$

converge weakly to a standard one-dimensional Brownian motion. In other words, if P_n are the laws of the random functions $Y_t^{(n)}$, then there is a probability measure P (the Wiener measure) over the space of real continuous functions on $[0,1]$, $C[0,1]$, fulfilling the properties from Definition 1.1 and such that $P_n(G) \to P(G)$ for any Borel set G of $C[0,1]$ with $P(\partial G) = 0$.

This result, which is also known as the Functional Central Limit Theorem, can be thought of an analogous of the Central Limit Theorem for random functions.

In this paper we will prove this result in the particular case where $\mathbb{P}\{X_1 = -1\} = \mathbb{P}\{X_1 = 1\} = q/2$, $\mathbb{P}\{X_1 = 0\} = 1 - q$ for $q \in (0,1)$, which is the case of a symmetric Lazy Random Walk, using the same techniques described in [1].

2. Preliminaries

Proving weak convergence in $C[0,1]$ or, in general, in any measurable space $(S,\mathcal{S})$, where S is a metric space and $\mathcal{S}$ is its Borel σ-algebra, can be very complicated. In this section we shall provide the tools that will be used to simplify this task. Let us first recall a characterization of the weak convergence of probability measures given in [4]:

Theorem 2.1. *Let $\{P_n\}_{n\in\mathbb{N}}$ and P be probability measures on $(S,\mathcal{S})$. Then the following statements are equivalent:*

(i) *P_n converges weakly to P,*

(ii) *$\int_S f\,dP_n \to \int_S f\,dP$ for any bounded and continuous function $f\colon S\to\mathbb{R}$,*

(iii) *$P_n(G)\to P(G)$ for any $G\in\mathcal{S}$ such that $P(\partial G)=0$.*

The collection of all the measures on $(S,\mathcal{S})$ can be thought of as a function space from the σ-algebra $\mathcal{S}$ to the interval $[0,1]$. Given this interpretation, and as we do with the space of real continuous functions, we can define a notion of relative compactness.

Definition 2.2. An arbitrary family of probability measures Π on $(S,\mathcal{S})$ is said to be relatively compact if for any sequence $\{P_n\}_{n\in\mathbb{N}}\subset\Pi$ exists a probability P on $(S,\mathcal{S})$ (not necessarily in Π) and a subsequence $\{P_{n_i}\}_{i\in\mathbb{N}}$ converging weakly to P.

The following theorem gives us a characterization of the weak convergence which will be crucial when proving the existence and uniqueness of probability measures on metric spaces:

Theorem 2.3. *Let $\{P_n\}_n$ and P be probabilities on $(S,\mathcal{S})$. Then P_n converges weakly to P if, and only if, every subsequence $\{P_{n_i}\}_i$ has a further subsequence $\{P_{n_{i_m}}\}_m$ converging weakly to P when $m\to\infty$.*

Proof. We will only prove the sufficiency, since it is the useful part. If P_n does not converge weakly to P, by Theorem 2.1, there is a bounded and continuous function $f\colon S\to\mathbb{R}$ such that $\int_S f\,dP_n \not\to \int_S f\,dP$, meaning that for some $\varepsilon>0$ and some subsequence $\{P_{n_i}\}_i$ we have $\left|\int_S f\,dP_{n_i}-\int_S f\,dP\right|>\varepsilon$ for all i. This in particular implies that no further subsequence can be weakly convergent to P. □

Continuing with the parallelism with the space of continuous functions, where the Arzelà–Ascoli Theorem gives a characterization of the relative compactness of a family of functions in terms of equicontinuity and pointwise boundedness, there might be a characterization of these relatively compact families of probability measures in terms of the compact sets of our metric space. In this case, the equivalence is given by Prohorov's Theorem.

Definition 2.4. An arbitrary family of probability measures Π on $(S,\mathcal{S})$ is said to be tight if for any $\varepsilon>0$ there is a compact subset K of S such that $P(K)>1-\varepsilon$ for any $P\in\Pi$.

The tightness of a family of probability measures tells us that there is no escape of mass to "infinity". In other words, all the mass is concentrated in our space S and not in any extension of it.

Theorem 2.5 (Prohorov)**.** *If an arbitrary family of probability measures Π on $(S,\mathcal{S})$ is tight, then it is relatively compact. Furthermore, if S is a polish space and Π is relatively compact, then it is tight.*

Two different proofs of this result can be found in [2] and [4].

Let us now see how we will apply these results in the particular case where $S = C := C[0,1]$ with the uniform metric. First of all, recall that C is a complete and separable metric space with this metric, implying that a family of probability measures on $(C, \mathcal{C})$, where $\mathcal{C}$ is the Borel σ-algebra of $C[0,1]$, is relatively compact if, and only if, it is tight.

Definition 2.6. The finite dimensional distributions of a probability measure on $(C, \mathcal{C})$ are the compositions $P\pi^{-1}_{t_1,\dots,t_k} := P \circ \pi^{-1}_{t_1,\dots,t_k}$, where $k \in \mathbb{N}$, $0 \leq t_1 < \cdots < t_k \leq 1$ and $\pi_{t_1,\dots,t_k}(f) = (f(t_1), \dots, f(t_k))$ for any $f \in C$.

It can be shown (see [2] or [4] and [3]) that if X is a random function (that is, a measurable function from a sample space with a certain σ-algebra to $(C, \mathcal{C})$), then the laws of the random vectors $(X_{t_1}, \dots, X_{t_k})$ (the finite dimensional random vectors of the stochastic process) coincide with the finite dimensional distributions of the law of the random function X and that the finite dimensional distributions of a probability measure determine unequivocally the probability measure. In other words, if P and Q are probabilities over $(C, \mathcal{C})$ such that $P\pi^{-1}_{t_1,\dots,t_k} = Q\pi^{-1}_{t_1,\dots,t_k}$ for any $k \in \mathbb{N}$ and any $0 \leq t_1 < \cdots < t_k \leq 1$, then $P = Q$.

Theorem 2.7. *If a sequence of probabilities $\{P_n\}_n$ is relatively compact, and if $P_n\pi^{-1}_{t_1,\dots,t_k}$ converges weakly to some probability measure $\mu_{t_1,\dots,t_k}$ on $(\mathbb{R}^k, \mathcal{R}^k)$ (being $\mathcal{R}^k$ the Borel σ-algebra of $\mathbb{R}^k$) for all $k \in \mathbb{N}$ and for any $0 \leq t_1 < \cdots < t_k \leq 1$, then some probability P on $(C, \mathcal{C})$ satisfies $P\pi^{-1}_{t_1,\dots,t_k} = \mu_{t_1,\dots,t_k}$ for all k and $t_1, \dots, t_k$ and P_n converges weakly to P.*

Proof. Indeed, let $\{P_{n_i}\}_i$ be any subsequence of $\{P_n\}_n$. Then some further subsequence $\{P_{n_{i_m}}\}_m$ converges weakly to a probability P. Now, since $\pi_{t_1,\dots,t_k} \colon C \to \mathbb{R}^k$ is a continuous function on the space C, by Lebesgue's change of variables formula, we have for any bounded and continuous function, $f \colon C \to \mathbb{R}$,

$$\int_{\mathbb{R}^k} f \cdot d(P_{n_{im}}\pi^{-1}_{t_1\cdots t_k}) = \int_C f \circ \pi_{t_1\cdots t_k} \cdot dP_{n_{im}} \xrightarrow{m\to\infty} \int_C f \circ \pi_{t_1\cdots t_k} \cdot dP = \int_{\mathbb{R}^k} f \cdot d(P\pi^{-1}_{t_1\cdots t_k}).$$

Meaning that $P_{n_{im}}\pi^{-1}_{t_1\cdots t_k}$ converges weakly to $P\pi^{-1}_{t_1\cdots t_k}$ for all k and $t_1, \dots, t_k$. By uniqueness of the weak limit of a sequence of probabilities and Theorem 2.3, we have $P\pi^{-1}_{t_1\cdots t_k} = \mu_{t_1,\dots,t_k}$.

To prove the second half of the theorem, if P is the probability measure found in the first half, we have that $P_n\pi^{-1}_{t_1\cdots t_k}$ converges weakly to $P\pi^{-1}_{t_1\cdots t_k}$ for all k and $t_1, \dots, t_k$. However, given that $\{P_n\}_n$ is relatively compact, any subsequence will have a further subsequence $\{P_{n_{im}}\}_m$ converging weakly to some probability Q on $(C, \mathcal{C})$. Again, by Lebesgue's change of variables formula and uniqueness of the limit, we conclude that $Q\pi^{-1}_{t_1\cdots t_k} = P\pi^{-1}_{t_1\cdots t_k}$ for all k and $t_1, \dots, t_k \in [0,1]$, meaning that $Q = P$. Namely, any subsequence has a further subsequence weakly convergent to P and thus, by Theorem 2.3, we see that P_n weakly converges to P. □

This last result tells us that, in order to prove the existence of the Wiener measure/process and convergence towards this stochastic process, one only has to construct a sequence of stochastic processes whose laws are relatively compact (or, by virtue of Prohorov's Theorem, tight) and then check that the finite dimensional vectors converge in law to the desired ones.

In general, it is easier to prove the tightness of a sequence of random functions (namely, the tightness of their laws) rather than its relative compactness. Billingsley's Criterion provides us a tool towards this direction.

Theorem 2.8 (Billingsley's Criterion). *Let $\{X^{(n)}\}_n$ be a sequence of random functions. Then it is tight if the following conditions are fulfilled:*

(i) $\{X_0^{(n)}\}$ *is tight (as a sequence of random variables).*

(ii) *For some $\gamma \geq 0$, $\alpha > 1$ and some continuous, non-decreasing function $F\colon [0,1] \to \mathbb{R}$, we have*

$$\mathbb{P}\{|X_t^{(n)} - X_s^{(n)}| \geq \lambda\} \leq \frac{1}{\lambda^\gamma}|F(t) - F(s)|^\alpha$$

for any $\lambda > 0$, $n \in \mathbb{N}$ and $s, t \in [0,1]$.

Due to Markov's inequality, if we manage to prove that $\mathbb{E}[|X_t^{(n)} - X_s^{(n)}|^\gamma] \leq |F(t) - F(s)|^\alpha$, condition (ii) of Billingsley's Criterion will be fulfilled.

A proof of this result can be found in [2].

3. Tightness and convergence of the finite dimensional distributions

Let us first prove the tightness of the sequence of random functions defined by Equation (1) for the particular case of the symmetric Lazy Random Walk (note that here we have $\mu = 0$ and $\sigma^2 = q$).

Theorem 3.1. *There is a positive constant C such that $\mathbb{E}[(Y_t^{(n)} - Y_s^{(n)})^4] \leq C(t-s)^2$ for any $n \in \mathbb{N}$ and any $s, t \in [0,1]$.*

Proof. If $s = t$, the result is trivial. Without any loss of generality, let us assume that $0 \leq s < t \leq 1$.

Let us first note that we can rewrite $Y_t^{(n)}$ as follows:

$$Y_t^{(n)} = \frac{1}{\sqrt{n}} \int_0^{nt} \theta(x)\, dx, \quad \theta(x) = \sum_{k=1}^{\infty} \frac{X_k}{\sqrt{q}} \mathbb{I}_{[k-1,k)}(x).$$

Indeed, the first integral in $\int_0^{nt} \theta(x)\, dx = \int_0^{[nt]} \theta(x)\, dx + \int_{[nt]}^{nt} \theta(x)\, dx$ reduces to the integral of the sum $\sum_{k=1}^{[nt]} X_k \mathbb{I}_{[k-1,k)}(x)/\sqrt{q}$ due to the contribution of characteristic functions whose interval $[k-1,k)$ lies in the interval $[0,[nt]]$, while in the second integral, this only occurs in the summand whose characteristic function lies in the interval $[[nt], nt]$. Thus,

$$\int_0^{[nt]} \theta(x)\, dx = \sum_{k=1}^{[nt]} \frac{X_k}{\sqrt{q}} \int_0^{[nt]} \mathbb{I}_{[k-1,k)}(x)\, dx = \sum_{k=1}^{[nt]} \frac{X_k}{\sqrt{q}},$$

$$\int_{[nt]}^{nt} \theta(x)\, dx = \frac{X_{[nt]+1}}{\sqrt{q}} \int_{[nt]}^{nt} dx = (nt - [nt]) \frac{X_{[nt]+1}}{\sqrt{q}}.$$

Now, given that $s < t$, we have that $\sqrt{n}(Y_t^{(n)} - Y_s^{(n)}) = \int_{[ns,nt]} \theta(x)\, dx$, from where we get

$$(Y_t^{(n)} - Y_s^{(n)})^4 = \frac{1}{n^2} \int_{[ns,nt]^4} \theta(x_1)\theta(x_2)\theta(x_3)\theta(x_4)\, d^4\mathbf{x}$$
$$= \frac{24}{n^2} \int_{[ns,nt]^4} \theta(x_1)\theta(x_2)\theta(x_3)\theta(x_4)\mathbb{I}_{\{x_1\le x_2\le x_3\le x_4\}}(\mathbf{x})\, d^4\mathbf{x},$$

where $\mathbf{x} = (x_1, x_2, x_3, x_4)$, $d^4\mathbf{x} = dx_1\, dx_2\, dx_3\, dx_4$ and in the last step we have used that the product $\theta(x_1)\theta(x_2)\theta(x_3)\theta(x_4)$ remains invariant under permutations of the variables $x_1, \dots, x_4$ when $\mathbf{x} \in [ns, nt]^4$ to fix an ordering by making use of the indicator function $\mathbb{I}_{\{x_1\le x_2\le x_3\le x_4\}}$.

Given that the sums $\theta(x)$ are always finite for $x \in [ns, nt]$, the function $\theta(x_1)\theta(x_2)\theta(x_3)\theta(x_4)$ is integrable and we can make use of Fubini's Theorem to say that

$$\mathbb{E}[(Y_t^{(n)} - Y_s^{(n)})^4] = \frac{24}{n^2} \int_{[ns,nt]^4} \mathbb{E}[\theta(x_1)\theta(x_2)\theta(x_3)\theta(x_4)\mathbb{I}_{\{x_1\le x_2\le x_3\le x_4\}}(\mathbf{x})]\, d^4\mathbf{x}.$$

On the other hand, we have that

$$\mathbb{E}[\theta(x_1)\theta(x_2)\theta(x_3)\theta(x_4)\mathbb{I}_{\{x_1\le x_2\le x_3\le x_4\}}(\mathbf{x})]$$
$$= \frac{1}{q^2} \sum_{k_1,k_2,k_3,k_4} \mathbb{E}[X_{k_1}X_{k_2}X_{k_3}X_{k_4}]\mathbb{I}_{[k_1-1,k_1)}(x_1) \cdot \ldots \cdot \mathbb{I}_{[k_4-1,k_4)}(x_4)\mathbb{I}_{\{x_1\le\cdots\le x_4\}}(\mathbf{x})$$
$$= \sum_{j,k} \left[\left(\frac{1}{q} - 1\right)\delta_{jk} + 1\right] \mathbb{I}_{[k-1,k)^2}(x_1, x_2) \cdot \mathbb{I}_{[j-1,j)^2}(x_3, x_4)\mathbb{I}_{\{x_1\le\cdots\le x_4\}}(\mathbf{x}).$$

Indeed, given that $\mathbb{E}[X_i] = 0$, $\mathbb{E}[X_i^2] = q$, $\mathbb{E}[X_i^4] = q$ and that the random variables $\{X_i\}_i$ are independent, we have that $\mathbb{E}[X_{k_1}X_{k_2}X_{k_3}X_{k_4}] = 0$ if there is a subscript k_j such that $k_j \ne k_i$ for all $i \in \{1, 2, 3, 4\}\backslash\{j\}$, $\mathbb{E}[X_{k_1}X_{k_2}X_{k_3}X_{k_4}] = q^2$ if $k_1 = k_2 \ne k_3 = k_4$ and $\mathbb{E}[X_{k_1}X_{k_2}X_{k_3}X_{k_4}] = q$ if $k_1 = k_2 = k_3 = k_4$. Any other possible cases are discarded due to the presence of the indicator function $\mathbb{I}_{\{x_1\le x_2\le x_3\le x_4\}}(\mathbf{x})$.

Now, making use of the inequality $\mathbb{I}_{\{x_1\le\cdots\le x_4\}}(\mathbf{x}) \le \mathbb{I}_{\{x_1\le x_2\}}(x_1, x_2) \cdot \mathbb{I}_{\{x_3\le x_4\}}(x_3, x_4)$, we see that

$$\begin{aligned}\mathbb{E}[(Y_t^{(n)} - Y_s^{(n)})^4]&\\ &\le \frac{24}{n^2} \int_{[ns,nt]^4} \sum_k \left(\frac{1}{q}-1\right) \mathbb{I}_{[k-1,k)^2}(x_1, x_2) \cdot \mathbb{I}_{[k-1,k)^2}(x_3, x_4)\mathbb{I}_{\{x_1\le x_2\}}(x_1, x_2) \cdot \mathbb{I}_{\{x_3\le x_4\}}(x_3, x_4)\, d^4\mathbf{x}\\ &\quad + \frac{24}{n^2} \int_{[ns,nt]^4} \sum_{j,k} \mathbb{I}_{[k-1,k)^2}(x_1, x_2) \cdot \mathbb{I}_{[j-1,j)^2}(x_3, x_4)\mathbb{I}_{\{x_1\le x_2\}}(x_1, x_2) \cdot \mathbb{I}_{\{x_3\le x_4\}}(x_3, x_4)\, d^4\mathbf{x}.\end{aligned} \tag{2}$$

Let us focus on the first term of the latter expression. After some simple manipulations we can rewrite this term as follows:

$$\frac{24}{n^2}\left(\frac{1}{q} - 1\right) \sum_k \left(\int_{[ns,nt]^2} \mathbb{I}_{[k-1,k)^2}(x_1, x_2) \cdot \mathbb{I}_{\{x_1\le x_2\}}(x_1, x_2)\, dx_1\, dx_2\right)^2$$
$$= 24n^2\left(\frac{1}{q} - 1\right) \sum_k \left(\int_{[s,t]^2} \mathbb{I}_{\left[\frac{k-1}{n},\frac{k}{n}\right)^2}(y_1, y_2) \cdot \mathbb{I}_{\{y_1\le y_2\}}(y_1, y_2)\, dy_1\, dy_2\right)^2,$$

where in the last step we have introduced the change of variables $y_i = x_i/n$. Now recall that if $\{a_k\}_k$ is a sequence of non-negative real numbers, then $\left(\sum_k a_k\right)^2 = \sum_k a_k^2 + \sum_{k\neq l} a_k a_l \geq \sum_k a_k^2$, from where we infer that the latter expression is lesser than

$$24n^2\left(\frac{1}{q}-1\right)\left(\int_{[s,t]^2}\sum_k\left(\mathbb{I}_{\left[\frac{k-1}{n},\frac{k}{n}\right)^2}(y_1,y_2)\right)\mathbb{I}_{\{y_1\leq y_2\}}(y_1,y_2)\,dy_1\,dy_2\right)^2$$
$$\leq 24n^2\left(\frac{1}{q}-1\right)\left(\int_{[s,t]^2}\mathbb{I}_{\left\{y_2-y_1\leq\frac{1}{n}\right\}}(y_1,y_2)\cdot\mathbb{I}_{\{y_1\leq y_2\}}(y_1,y_2)\,dy_1\,dy_2\right)^2.$$

Where we have used that $\sum_k\left(\mathbb{I}_{\left[\frac{k-1}{n},\frac{k}{n}\right)^2}(y_1,y_2)\right) \leq \mathbb{I}_{\left\{y_2-y_1\leq\frac{1}{n}\right\}}$. Lastly, we have that

$$\int_s^t\int_s^{y_2}\mathbb{I}_{\left\{y_2-y_1\leq\frac{1}{n}\right\}}(y_1,y_2)\,dy_1\,dy_2 = \int_s^t\int_{\max\{y_2-1/n,s\}}^{y_2}dy_1\,dy_2 \leq \int_s^t\int_{y_2-1/n}^{y_2}dy_1\,dy_2 = \frac{t-s}{n},$$

which allows us to conclude that the first term in Equation (2) can be bounded by $24\left(\frac{1}{q}-1\right)(t-s)^2$.

Proceeding in a similar manner, we can see that the second term in Equation (2) can be bounded by $24(t-s)^2$ and, all in all, we conclude that

$$\mathbb{E}[(Y_t^{(n)}-Y_s^{(n)})^4] \leq \frac{24}{q}(t-s)^2. \qquad \square$$

With this (and Billingsley's Criterion) we have verified that the sequence of random functions defined by Equation (1) is tight (the sequence of random variables $Y_0^{(n)}$ is tight since it is identically zero for all n). We now see that the finite dimensional vectors of the sequence converge to the desired ones:

Theorem 3.2. *For any $k \in \mathbb{N}$ and any $0 \leq t_1 < \cdots < t_k \leq 1$, the random vectors $(Y_{t_1}^{(n)},\ldots,Y_{t_k}^{(n)})$ converge in law to the random vectors $(B_{t_1},\ldots,B_{t_k})$ when $n\to\infty$ and where the random variables B_{t_j} are normally distributed with null mean and variance t_j (with $B_{t_j} = 0$ if $t_j = 0$) and are such that $B_{t_1}, B_{t_2}-B_{t_1},\ldots,B_{t_k}-B_{t_{k-1}}$ are independent and therefore (due to the change of variables formula) the random variables $B_{t_{j+1}}-B_{t_j}$ are normally distributed with zero mean and variance $t_{j+1}-t_j$.*

Proof. Before starting to prove the statement, lets first recall the following facts:

(i) If $\{Z_n\}_n$ and $\{W_n\}_n$ are sequences of random vectors such that for every $\varepsilon > 0$

$$\mathbb{P}\{\|Z_n - Y_n\| > \varepsilon\} \xrightarrow{n\to\infty} 0,$$

then, if W_n converges in law to a certain random vector W, then so does Z_n.

(ii) If a sequence of random vectors $\{Z_n\}$ in $\mathbb{R}^k$ converges in law to a certain random vector Z and $h\colon \mathbb{R}^k \to \mathbb{R}^{k'}$ is a continuous function, then $h(Z_n)$ converges in law to $h(Z)$. In addition, if h is invertible and its inverse is continuous, then Z_n converges in law to Z if, and only if, $h(Z_n)$ converges in law to $h(Z)$.

(iii) A sequence of random vectors $\{Z_n\}$ in $\mathbb{R}^k$ with characteristic functions φ_n converges in law to a certain random vector Z with characteristic function φ if, and only if, $\varphi_n(\mathbf{u}) \to \varphi(\mathbf{u})$ for every $\mathbf{u}\in\mathbb{R}^k$ when $n\to\infty$.

Now, for every $t \in [0,1]$, given that $nt - [nt] \in [0,1)$, we have that

$$\left| Y_t^{(n)} - \frac{1}{\sqrt{n}} S_{[nt]} \right| \leq \frac{|X_{[nt]+1}|}{\sqrt{nq}}.$$

Meaning that, by Chebyshev's inequality and for every $\varepsilon > 0$,

$$\mathbb{P}\left\{ \left| Y_t^{(n)} - \frac{1}{\sqrt{n}} S_{[nt]} \right| > \varepsilon \right\} \leq \mathbb{P}\{|X_{[nt]+1}| > \varepsilon\sqrt{nq}\} \leq \frac{\mathrm{Var}(X_{[nt]+1})}{\varepsilon^2 nq} = \frac{1}{n\varepsilon^2} \xrightarrow{n\to\infty} 0.$$

Which implies, setting $Y_{n,k} = (Y_{t_1}^{(n)}, \dots, Y_{t_k}^{(n)})$ and $S_{n,k} = (S_{[t_1 n]}, \dots, S_{[t_k n]})$,

$$\mathbb{P}\left\{ \left\| Y_{n,k} - \frac{1}{\sqrt{n}} S_{n,k} \right\| > \varepsilon \right\} = \mathbb{P}\left\{ \left\| Y_{n,k} - \frac{1}{\sqrt{n}} S_{n,k} \right\|^2 > \varepsilon^2 \right\} \leq \sum_{j=1}^{k} \mathbb{P}\left\{ \left| Y_{t_j}^{(n)} - \frac{1}{\sqrt{n}} S_{[nt_j]} \right| > \frac{\varepsilon}{\sqrt{k}} \right\}$$

and this last quantity goes to zero as n approaches infinity for every $\varepsilon > 0$.

By virtue of the first two facts mentioned before, we shall prove that the random vectors $S^{(n)} = (S_{[t_1 n]}, S_{[t_2 n]} - S_{[t_1 n]}, \dots, S_{[t_k n]} - S_{[t_{k-1} n]})/\sqrt{n}$ converge in law to a random vector $(B_{t_1}, \dots, B_{t_k} - B_{t_{k-1}})$ whose components verify the desired properties.

Now, if $t_0 = 0 < t_1$ (if $t_1 = 0$, we can omit this step and proceed in a similar way), note that

$$\frac{1}{\sqrt{n}}(S_{[t_{l+1} n]} - S_{[t_l n]}) = \frac{1}{\sqrt{nq}} \sum_{j=[t_l n]+1}^{[t_{l+1} n]} X_j$$

for all $l \in 0, \dots, k-1$. Since the random variables $\{X_l\}_l$ are independent, this means that the components of the vector $S^{(n)}$ are independent and thus, we only need to prove that each component $(S_{[t_{l+1} n]} - S_{[t_l n]})/\sqrt{n}$ converges in law to a normal random variable with null mean and variance $t_{l+1} - t_l$ (recall that the characteristic function of a vector whose components are independent is the product of the characteristic functions of each component and therefore the third fact can be applied). If we manage to prove that

$$\mathbb{P}\left\{ \left| \frac{1}{\sqrt{nq}} \sum_{j=[t_l n]+1}^{[t_{l+1} n]} X_j - \frac{\sqrt{t_{l+1} - t_l}}{\sqrt{q[n(t_{l+1} - t_l)]}} \sum_{j=1}^{[n(t_{l+1} - t_l)]} X_j \right| > \varepsilon \right\} \xrightarrow{n\to\infty} 0 \tag{3}$$

for every $\varepsilon > 0$, then, due to the first fact, we will have proven that the two sums (multiplied by their respective factors) will have the same limit in law. But, due to the Central Limit Theorem, the sum

$$\frac{\sqrt{t_{l+1} - t_l}}{\sqrt{q[n(t_{l+1} - t_l)]}} \sum_{j=1}^{[n(t_{l+1} - t_l)]} X_j$$

converges in law to a normal random variable with zero mean and variance $t_{l+1} - t_l$, concluding the proof.

To verify identity (3), we first assume that $[n(t_{l+1} - t_l)] < [nt_l] + 1$. If this is the case, then the random variables X_j involved in Equation (3) are all independent and, by Chebyshev's inequality,

$$\mathbb{P}\left\{ \left| \frac{1}{\sqrt{nq}} \sum_{j=[t_l n]+1}^{[t_{l+1} n]} X_j - \frac{\sqrt{t_{l+1} - t_l}}{\sqrt{q[n(t_{l+1} - t_l)]}} \sum_{j=1}^{[n(t_{l+1} - t_l)]} X_j \right| > \varepsilon \right\} \leq \frac{1}{\varepsilon^2} \left[\frac{[nt_{l+1}] - [nt_l]}{n} - (t_{l+1} - t_l) \right].$$

Using that $\lim_{n\to\infty} n\cdot s/[n\cdot s] = 1$ for all fixed $s>0$, we see that this last quantity goes to zero as n approaches infinity for every $\varepsilon > 0$.

If $[n(t_{l+1}-t_l)] \geq [nt_l]+1$, we first rewrite the difference in the probability (3) as follows:

$$\frac{1}{\sqrt{nq}}\sum_{j=[t_l n]+1}^{[t_{l+1}n]} X_j - \frac{\sqrt{t_{l+1}-t_l}}{\sqrt{q[n(t_{l+1}-t_l)]}}\sum_{j=1}^{[n(t_{l+1}-t_l)]} X_j$$
$$= \left(\frac{1}{\sqrt{nq}} - \frac{\sqrt{t_{l+1}-t_l}}{\sqrt{q[n(t_{l+1}-t_l)]}}\right)\sum_{j=[nt_l]+1}^{[n(t_{l+1}-t_l)]} X_j + \frac{1}{\sqrt{nq}}\sum_{j=[n(t_{l+1}-t_l)]+1}^{[nt_{l+1}]} X_j - \frac{\sqrt{t_{l+1}-t_l}}{\sqrt{q[n(t_{l+1}-t_l)]}}\sum_{j=1}^{[nt_l]} X_j.$$

Again, the random variables X_j involved are independent and thus, by Chebyshev's inequality,

$$\mathbb{P}\left\{\left|\frac{1}{\sqrt{nq}}\sum_{j=[t_l n]+1}^{[t_{l+1}n]} X_j - \frac{\sqrt{t_{l+1}-t_l}}{\sqrt{q[n(t_{l+1}-t_l)]}}\sum_{j=1}^{[n(t_{l+1}-t_l)]} X_j\right| > \varepsilon\right\}$$
$$\leq \frac{1}{\varepsilon^2}\left[\left(\frac{1}{\sqrt{n}} - \frac{\sqrt{t_{l+1}-t_l}}{\sqrt{[n(t_{l+1}-t_l)]}}\right)^2([n(t_{l+1}-t_l)]-[nt_l]) + \frac{[nt_{l+1}]-[n(t_{l+1}-t_l)]}{n} - \frac{[nt_l](t_{l+1}-t_l)}{[n(t_{l+1}-t_l)]}\right].$$

And this last quantity tends to zero as n tends to infinity for every $\varepsilon > 0$. □

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