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Dynamics of a family of meromorphic functions

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Resum *(CAT)*
En aquest projecte estudiem el comportament dinàmic de la família de funcions transcendents meromorfes $f_\lambda(z) = \lambda\left(\frac{e^z}{z+1} - 1\right)$, la qual es pot veure com l'anàleg meromorf de la ben coneguda família de polinomis cúbics de Milnor $P_a(z) = z^2(z-a)$ [12] o la seva versió entera $\lambda z^2 e^z$ [7, 8]. Contràriament a aquests dos casos, les conques d'atracció de f_λ no són simplement connexes. De fet, en aquest document es demostra que sota certes condicions, la conca d'atracció de $z = 0$ és infinitament connexa.

Abstract *(ENG)*
In this paper we analyze the dynamical behavior of the family of transcendental meromorphic maps $f_\lambda(z) = \lambda\left(\frac{e^z}{z+1} - 1\right)$. This family is the meromorphic analogue of the well-known Milnor family of cubic polynomials $P_a(z) = z^2(z-a)$ [12] or its entire version $\lambda z^2 e^z$ [7, 8]. Opposed to these two cases, the basins of attraction of f_λ are not simply connected. In fact, we prove that under certain conditions, the basin of attraction of $z = 0$ is infinitely connected.

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1. Introduction

In this work we focus on some dynamical aspects of transcendental meromorphic functions, i.e., we study the dynamical systems given by the iterates of meromorphic functions $f\colon \mathbb{C} \to \mathbb{C}_\infty$ with an essential singularity at ∞, where $\mathbb{C}_\infty$ denotes $\mathbb{C} \cup \{\infty\}$ or the Riemann sphere. Here the n-th iterate of a point $z \in \mathbb{C}$ is denoted by $f^n(z) = (f \circ \overset{(n)}{\cdots} \circ f)(z)$, and the sequence of iterates $\{f^n(z)\}_{n\in\mathbb{N}}$ is well-defined for all $z \in \mathbb{C}$ except for the countable set of poles and prepols of f of any order.

The interest for these functions is twofold: The essential singularity, on the one hand, adds a lot of chaos to the dynamical system, mainly because of Picard's Theorem, which states that in each punctured neighborhood of ∞, these functions assume each value of the Riemann sphere $\mathbb{C}_\infty$, with at most two exceptions (such exceptional values are known as omitted values), infinitely often. Hence, given a point $z \in \mathbb{C}$, if its orbit $\mathcal{O}_f^+(z) = \{f^n(z) : n \in \mathbb{N}\}$ is near ∞ at some moment, after one iteration it can land at almost any place of the plane. On the other hand, the presence of poles allows for more generality, when compared to entire functions, since ∞ is not required to be an omitted value.

The phase space (also called dynamical plane) of a meromorphic function f splits into two completely invariant sets: The Fatou set $F(f)$, which is the set of points $z \in \mathbb{C}$ such that the sequence of iterates $\{f^n\}_{n\in\mathbb{N}}$ is defined and normal in some neighborhood of z; and its complement, the Julia set $J(f)$.

It follows trivially from the definition that the Fatou set is open and hence the Julia set is closed. The first consists of components known as Fatou components, each of them might be either simply or multiply connected (including the infinitely connected case as we will see here). Let $U = U_0$ be a Fatou component, then $f^n(U)$ is contained in another component of $F(f)$ that we denote by U_n. We say that U_0 is preperiodic if $U_n = U_m$ for some $n > m \geq 0$ (if $m = 0$, we say that its periodic and if $n = 1$, we say that it is fixed or forward invariant), otherwise we say that U is wandering. Periodic Fatou components are classified according to the following celebrated result of Fatou [2], which for simplicity we state for fixed components.

Theorem 1.1 (Classification Theorem for fixed Fatou components)**.** *Let U be a fixed Fatou component. Then we have one of the following possibilities:*

(i) *U contains an attracting fixed point z_0 and $f^n(z) \xrightarrow[n\to\infty]{} z_0$ for all $z \in U$, which is called the* immediate attractive basin *of z_0.*

(ii) *∂U contains a fixed point z_0 and $f^n(z) \xrightarrow[n\to\infty]{} z_0$ for all $z \in U$. Moreover, $f'(z_0) = 1$ if $z_0 \in \mathbb{C}$ and U is called a* parabolic (or Leau) domain.

(iii) *There exists $\phi\colon U \to \mathbb{D}$ conformal such that $\phi(f(\phi^{-1}(z))) = e^{2\pi i\alpha}z$ for some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Moreover, U is called a* Siegel disk.

(iv) *There exists $\phi\colon U \to A$ conformal where $A = \{z : 1 < |z| < r\}$, $r > 1$, is an annulus such that $\phi(f(\phi^{-1}(z))) = e^{2\pi i\alpha}z$ for some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Moreover, U is called a* Herman ring.

(v) *There exists $z_0 \in \partial U$ such that $f^n(z) \xrightarrow[n\to\infty]{} z_0$ for all $z \in U$ but $f(z_0)$ is not defined. Moreover, U is called a* Baker domain.

In order to study the Fatou components we introduce the notion of the singularities of the inverse, which are the points $a \in \mathbb{C}$ where some branch of f^{-1} is not well-defined (holomorphic and injective) in a neighborhood of $a \in \mathbb{C}$. Two different cases can arise: either there exists $z \in \mathbb{C}$ such that $f(z) = a$ and $f'(z) = 0$ ($z \in \mathbb{C}$ is then said to be a critical point and $a \in \mathbb{C}$ a critical value); or there exists a curve $\gamma\colon [0, \infty) \to \mathbb{C}$ such that $\gamma(t) \xrightarrow[t\to\infty]{} \infty$ and $f(\gamma(t)) \xrightarrow[t\to\infty]{} a$ ($a \in \mathbb{C}$ is then said to be an asymptotic value and the curve γ an asymptotic path).

Singular values (critical or asymptotic) play a fundamental role in the dynamical behavior of holomorphic (or meromorphic) functions. For example, any immediate attractive or parabolic basin of attraction needs to contain a singular value. In the remaining cases they are also relevant (see [1, 2, 4, 5, 11]), but in this paper we will focus in the study of an attractive basin of the family of maps

$$f_\lambda(z) = \lambda\left(\frac{e^z}{z+1} - 1\right),$$

where $\lambda \in \mathbb{C} \setminus \{0\}$ is a complex parameter.

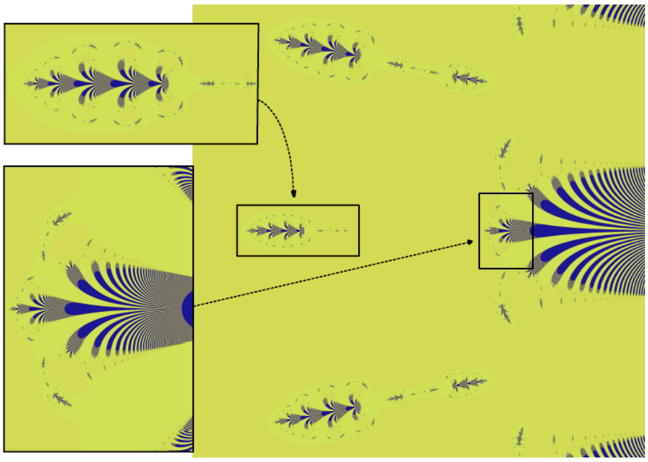

Figure 1: In green, $F(f_{0.89})$. Range $(-5, 7) \times (-6, 6)$.

Maps in this family are the simplest meromorphic maps with two singular values: $z = 0$ which is a fixed critical value (and fixed point), and $z = -\lambda$, which is an asymptotic value whose orbit depends on λ. It has also one single pole $z = -1$, which is not omitted except for $\lambda = 1$. Since $z = 0$ is a superattracting fixed point (i.e., a critical point which is a fixed point), its basin of attraction $\mathcal{A}_\lambda(0)$ is non-empty for all values of λ.

This family can be viewed as the meromorphic analogue to the well-known Milnor family of cubic polynomials $P_a(z) = z^2(z - a)$ [12] or its entire version $\lambda z^2 e^z$ [7, 8], both having also a superattracting fixed point and a free second singular value, which may or may not be captured by the attracting basin of 0. In both cases all components of the Fatou set are simply connected. In contrast, in this paper we prove that the basin of attraction of $z = 0$ for f_λ is infinitely connected for some parameter values.

Additionally, it is well-known that functions with only finitely many singular values do not have Wandering nor Baker domains [3, 4, 6, 10], hence $F(f_\lambda)$ does not have any of these components. Moreover, since any attractive basin or rotation domain needs a singular value, we can have at most two periodic cycles of Fatou components for every parameter value, one of which is always the basin of $z = 0$.

Hence it is to our interest to study the main capture component $\mathcal{C}_0 = \{\lambda \in \mathbb{C}^* : -\lambda \in \mathcal{A}^*_\lambda(0)\}$, where $\mathcal{A}^*_\lambda(0)$ denotes the immediate basin of attraction of $z = 0$. In this case there is only one Fatou component and we can draw an accurate picture of $F(f)$ by considering the points whose orbit is attracted to $z = 0$.

After addressing the study of the dynamical properties of f_λ for $\lambda \in \mathcal{C}_0$, we prove:

Theorem A. *If $-\lambda \in \mathcal{A}^*_\lambda(0)$, then $\mathcal{A}_\lambda(0)$ is connected and infinitely connected. Moreover, the set $\mathcal{C}_0 := \{\lambda \in \mathbb{C}^* : -\lambda \in \mathcal{A}^*_\lambda(0)\}$ contains a punctured disk of center 0 and radius $1/2$ (see Figure 2).*

The paper is structured as follows. In Section 2 we prove some estimates which are useful in Section 3, where we prove Theorem A.

2. Preliminaries

Our goal in this section is to prove some estimates that will be useful in the later section. We start by estimating the radius of the largest disk contained in $\mathcal{A}_\lambda(0)$, which we denote by

$$r_\lambda = \sup\{r > 0 : D(0, r) \subset \mathcal{A}_\lambda(0)\} < 1,$$

where the last inequality follows from the fact that $z = -1$ is a pole and hence belongs to the Julia set.

Proposition 2.1 (Maximum inner disk)**.** *For every $\lambda \in \mathbb{C}^*$,*

$$r_\lambda \geq \varepsilon(\lambda) := \frac{1}{2}\left(2 + |\lambda| - \sqrt{|\lambda|^2 + 4|\lambda|}\right) \in (0, 1).$$

Consequently $D(0, \varepsilon(\lambda)) \subset \mathcal{A}_\lambda(0)$.

Proof. For $0 < \varepsilon < 1$ and $|z| < \varepsilon$ we have

$$|f_\lambda(z)| = |f_\lambda(z) - f_\lambda(0)| \leq |\lambda| \left(\max_{|z|=\varepsilon}\left|\frac{ze^z}{(z+1)^2}\right|\right)|z|,$$

where we have used the Maximum Modulus Principle for f'_λ. For $z = \varepsilon e^{i\theta}$ we have

$$g_\lambda(\varepsilon, \theta) := |\lambda| \left|\frac{ze^z}{(z+1)^2}\right| = |\lambda| \frac{\varepsilon e^{\varepsilon\cos(\theta)}}{1 + \varepsilon^2 + 2\varepsilon\cos(\theta)}.$$

The goal is to obtain the maximum ε such that f_λ is a strict contraction in $D(0, \varepsilon)$, because then all points in $D(0, \varepsilon)$ converge to $z = 0$ under iteration, i.e., we want to obtain $\sup\{\varepsilon \in (0, 1) : g_\lambda(\varepsilon, \theta) < 1,\ \theta \in [0, 2\pi)\}$ since then it follows that $|f_\lambda(z)| < |z|$. We split it in two cases depending on θ.

- For $\theta \in [-\pi/2, \pi/2)$, we have $g_\lambda(\varepsilon, \theta) \leq |\lambda|\varepsilon e^\varepsilon/(1 + \varepsilon^2) =: g_{\lambda,1}(\varepsilon)$.
- For $\theta \in [\pi/2, 3\pi/2)$, we have $g_\lambda(\varepsilon, \theta) \leq |\lambda|\varepsilon/(1 - \varepsilon)^2 =: g_{\lambda,2}(\varepsilon)$.

Observe now that for $0 < \varepsilon < 1$, we always have $g_{\lambda,1}(\varepsilon) \leq g_{\lambda,2}(\varepsilon)$. Moreover, for $0 < \varepsilon < 1$,

$$|\lambda| \frac{\varepsilon}{(1 - \varepsilon)^2} < 1 \iff \varepsilon^2 - (2 + |\lambda|)\varepsilon + 1 > 0,$$

and this last polynomial has roots

$$\varepsilon(\lambda) = \frac{1}{2}\left(2 + |\lambda| - \sqrt{|\lambda|^2 + 4|\lambda|}\right) \quad \text{and} \quad \frac{2 + |\lambda| + \sqrt{|\lambda|(|\lambda| + 4)}}{2}.$$

Then $\varepsilon(\lambda) \in (0, 1)$ and the result follows. $\square$

As a consequence of Proposition 2.1 we see now that for a disk of parameters of definite size, the free asymptotic value $z = -\lambda$ belongs to the immediate basin of $z = 0$. The set of parameters with this property is called the main capture component $\mathcal{C}_0$, which was defined in the introduction.

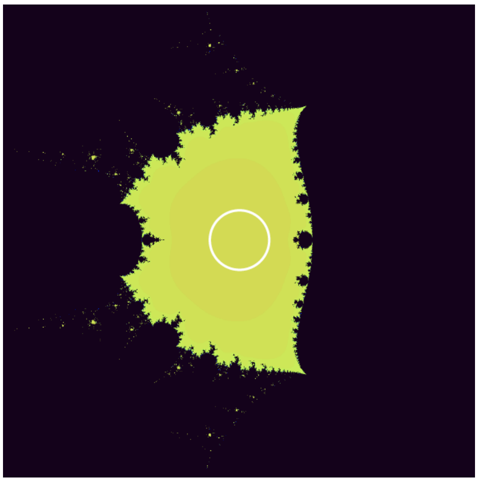

Figure 2: In green, $\mathcal{C}_0$. In white, $\partial D(0, 1/2)$. Range $(-4, 4) \times (-4, 4)$.

Corollary 2.2. $D^*(0, 1/2) = D(0, 1/2) \setminus \{0\} \subset \mathcal{C}_0$.

Proof. From the lower bound on r_λ given by Proposition 2.1, we obtain that $-\lambda \in D(0, \varepsilon(\lambda))$ if $\varepsilon(\lambda) - \lambda > 0$, or equivalently if

$$2 - |\lambda| > \sqrt{|\lambda|^2 + 4|\lambda|}.$$

It is easy to verify that this inequality holds for $|\lambda| < 1/2$. □

3. Connectivity of the basin of $z = 0$: Proof of Theorem A

We prove Theorem A in two parts. Assume in what follows that the asymptotic value $z = -\lambda$ belongs to the immediate basin of attraction of $z = 0$, or equivalently the parameter $\lambda \in \mathcal{C}_0$. We first show that the basin of attraction of $z = 0$ is connected, that is, $\mathcal{A}_\lambda(0) = \mathcal{A}^*_\lambda(0)$, and hence totally invariant (Theorem 3.2). Then we prove that under the same hypothesis, $\mathcal{A}^*_\lambda(0)$ is infinitely connected (Theorem 3.4).

Both results follow from two technical lemmas.

Lemma 3.1. *Let $\lambda \in \mathcal{C}_0$. Then, all asymptotic paths of $z = -\lambda$ intersect the same Fatou component of $F(f_\lambda)$.*

Proof. Given an asymptotic path Γ of $z=-\lambda$, i.e.,

$$\Gamma(t) \xrightarrow[t\to\infty]{} \infty \quad \text{and} \quad f_\lambda(\Gamma(t)) \xrightarrow[t\to\infty]{} -\lambda$$

then $\mathrm{Re}(\Gamma(t))$ must be bounded from above, i.e., exists $M_\Gamma < \infty$ such that

$$\sup_{t\geq 0} \mathrm{Re}(\Gamma(t)) \leq M_\Gamma.$$

Since $-\lambda \in \mathcal{A}^*_\lambda(0)$, there exists $\varepsilon > 0$ such that $D(-\lambda,\varepsilon) \subset \mathcal{A}^*_\lambda(0)$. Now, the preimages of $D(-\lambda,\varepsilon)$ must belong to $\mathcal{A}_\lambda(0)$. In particular, there exists $\nu < 0$ and a half-plane, $\Pi_\nu = \{z \in \mathbb{C} : \mathrm{Re}(z) < \nu\}$ such that $f_\lambda(\Pi_\nu) \subset D(-\lambda,\varepsilon)$, and hence Π_ν belongs to one component of $\mathcal{A}_\lambda(0)$. But now, all asymptotic paths must have unbounded negative real part and hence they must all intersect Π_ν. □

We now can prove that in this case the basin of $z=0$ is connected.

Theorem 3.2. *If $\lambda \in \mathcal{C}_0$, then $\mathcal{A}_\lambda(0) = \mathcal{A}^*_\lambda(0)$ is connected. In particular, $\mathcal{A}_\lambda(0)$ is totally invariant and is the whole Fatou set.*

Proof. Suppose that $-\lambda \in \mathcal{A}^*_\lambda(0)$. From Proposition 2.1, we can consider the disk $U_0 = D(0,\varepsilon(\lambda)) \subset \mathcal{A}^*_\lambda(0)$.

Now we pull-back U_0 in order to obtain the whole immediate basin $\mathcal{A}^*_\lambda(0)$:

Consider, for $N>0$, U_N as the connected component of $f_\lambda^{-1}(U_{N-1})$ that contains U_{N-1}. This recurrence defines a sequence of subsets $\{U_N\}_{N\geq 0}$ such that:

- $U_N \subset \mathcal{A}^*_\lambda(0)$ for all $N \geq 0$.
- $U_N \subset U_{N+1}$ for all $N \geq 0$.
- $\mathcal{A}^*_\lambda(0) = \bigcup_{N\geq 0} U_N$.

Since $-\lambda \in \mathcal{A}^*_\lambda(0)$, there exists $N>0$ such that $-\lambda \in U_N$ (i.e., $f_\lambda^N(-\lambda) \in U_0$), and we can find a path $\gamma \subset U_N$ that joins $-\lambda$ and 0.

So U_{N+1} is unbounded, because $z=-\lambda$ is an asymptotic value (a Picard Value), hence the preimage of γ must contain a path that joins 0 and ∞ (which is contained in U_{N+1}). Using Lemma 3.1 we obtain that, in fact, when $-\lambda \in \mathcal{A}^*_\lambda(0)$ all asymptotic tracts intersect $\mathcal{A}^*_\lambda(0)$.

Now suppose that $\mathcal{A}_\lambda(0)$ is not connected, then we must have at least two connected components, $\mathcal{A}^*_\lambda(0)$ and U. Furthermore,

$$f_\lambda(U) = \mathcal{A}^*_\lambda(0) \setminus \{-\lambda\}.$$

So U must contain a tail of an asymptotic path, but by Lemma 3.1 and the previous observation, this tail must be contained in $\mathcal{A}^*_\lambda(0)$ and the claim follows. □

Our next goal is to prove that $\mathcal{A}^*_\lambda(0)$ is infinitely connected (Theorem 3.4). To that end we first construct a closed curve in $\mathcal{A}^*_\lambda(0)$ which surrounds the pole $z=-1$. The Böttcher coordinates are the key ingredient. Given a closed curve γ, we denote by $\mathrm{ind}(\gamma,p)$ the winding number of γ with respect to the point $p \in \mathbb{C}$.

Lemma 3.3. *Let $-\lambda \in \mathcal{A}^*_\lambda(0)$. Then there exists a closed, simple curve β, contained in $\mathcal{A}_\lambda(0)$, such that $0 \notin \beta$ and $\mathrm{ind}(f_\lambda(\beta),0) = -1$.*

Proof. Let U be a neighborhood of $z = 0$ and $\varphi\colon U \to D(0,r)$ be the Böttcher map which locally conjugates f_λ to $Q_0(w) = w^2$.

Consider $\varepsilon < r < 1$ and define, $D = \varphi^{-1}(D(0,\varepsilon))$ and $D' = \varphi^{-1}(D(0,\varepsilon^2))$. The curves, $\tilde{r}_1(t) = i\sqrt{t}$, $\tilde{r}_2(t) = -i\sqrt{t}$, for $t \in [0,\varepsilon)$, are mapped by Q_0 to $\tilde{r}_0(t) = -t = Q_0(\tilde{r}_j(t))$, $j = 1,2$. Now set $r_j(t) = \varphi^{-1}(\tilde{r}_j(t))$, $j = 1,2$.

Since $-\lambda \in \mathcal{A}^*_\lambda(0)$, there exists a disk V centered at $z = -\lambda$ such that $\bar{V} \subset \mathcal{A}^*_\lambda(0)$ and, by Lemma 3.1, $f_\lambda^{-1}(\bar{V})$ contains a half-plane $\{z \in \mathbb{C} : \mathrm{Re}(z) < \nu\}$.

Furthermore, since by Theorem 3.2, $\mathcal{A}^*_\lambda(0) = \mathcal{A}_\lambda(0)$ is connected, we can find a simple curve $\alpha_0 \subset \mathcal{A}^*_\lambda(0)$ such that $\alpha_0(0) = 0$, $\alpha_0(1) = -\lambda$ and $(r_0)_{|[0,\varepsilon]} = (\alpha_0)_{|[0,\varepsilon]}$.

Define $s \in (\varepsilon, 1)$ such that $\alpha_0(s) \in \partial V$. Observe that the preimage of α_0 by f_λ are two simple curves, α_1, α_2 (because the preimage of $\tilde{r}_0$ by Q_0 consists of two disjoint curves), which are asymptotic paths, such that:

- $(r_j)_{|[0,\varepsilon]} = (\alpha_j)_{|[0,\varepsilon]}$ for $j = 1,2$.
- $(\alpha_1)_{|[\varepsilon,s)} \cap (\alpha_2)_{|[\varepsilon,s)} = \emptyset$, that is, because $0 \notin f_\lambda((\alpha_1)_{|[\varepsilon,s)}) = f_\lambda((\alpha_2)_{|[\varepsilon,s)}) = (\alpha_0)_{|[\varepsilon,s)}$ and hence, f_λ is conformal for every $z \in (\alpha_1)_{|[\varepsilon,s)} \cup (\alpha_2)_{|[\varepsilon,s)}$.

Now define.

- $\gamma_j = (\alpha_j)_{|[\varepsilon,s]}$ for $j = 0,1,2$.
- $\gamma_3 \subset f_\lambda^{-1}(\partial V)$ the simple curve that joins $\gamma_1(s)$ and $\gamma_2(s)$.
- $\tilde{\gamma}_4(t) = \varepsilon e^{-2\pi i t}$ and $\gamma_{4,1} = \varphi^{-1}((\tilde{\gamma}_4)_{|[1/4,3/4]})$, $\gamma_{4,2} = \varphi^{-1}((\tilde{\gamma}_4)_{|[-1/4,1/4]})$.

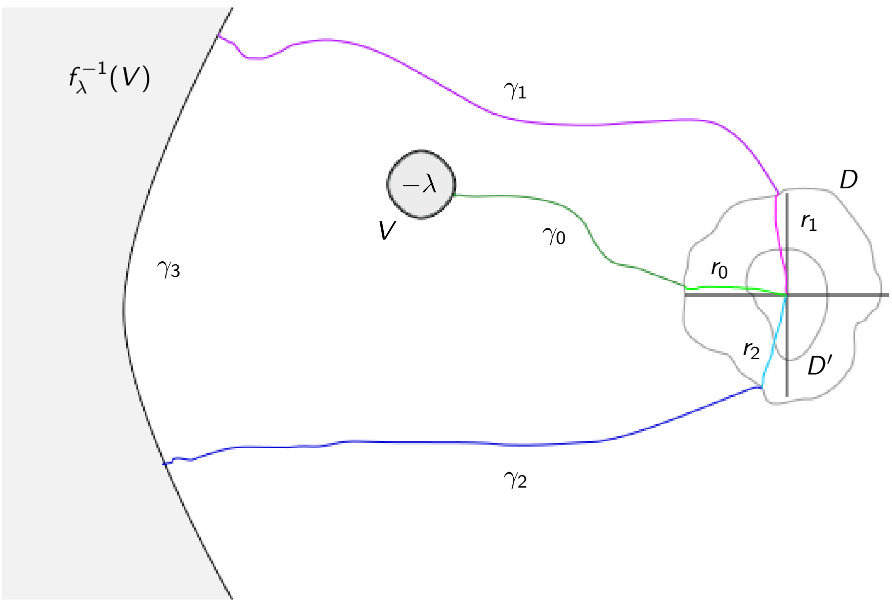

Figure 3: Representation of the curves and domains in the proof of Lemma 3.3.

Then we can define the curves,

$$\beta_1 = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_{4,1} \quad \text{and} \quad \beta_2 = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_{4,2},$$

which by construction are closed, simple curves contained in $\mathcal{A}_\lambda(0)$ that omit $z = 0$. Observe that β_1 (resp. β_2) can be parametrized as a simple closed curve preserving the orientation that $\gamma_{4,1}$ (resp. $\gamma_{4,2}$) inherits from $\tilde{\gamma}_4$.

Finally,

$$f_\lambda(\beta_j) = \partial V \cup \gamma_0 \cup \partial D', \quad j = 1, 2.$$

Hence, since $\partial V \cup \gamma_0$ does not contribute to $\text{ind}(f_\lambda(\beta_j), 0)$, we have $\text{ind}(f_\lambda(\beta_j), 0) = \text{ind}(\partial D', 0)$, thus

$$\text{ind}(f_\lambda(\beta_1), 0) = \text{ind}(\partial D', 0) = -1 \quad \text{or} \quad \text{ind}(f_\lambda(\beta_2), 0) = \text{ind}(\partial D', 0) = -1$$

(we want the curve β_1 or β_2 to be oriented counterclockwise), so we can take $\beta = \beta_1$ or $\beta = \beta_2$ so that $\text{ind}(f_\lambda(\beta), 0) = -1$. □

Finally, we prove the remaining part of the theorem.

Theorem 3.4. *If $\lambda \in \mathcal{C}_0 = \{\lambda \in \mathbb{C}^* : -\lambda \in \mathcal{A}^*_\lambda(0)\}$, then $\mathcal{A}_\lambda(0) = F(f_\lambda)$ is infinitely connected.*

Proof. By Theorem 3.2 we know that $\mathcal{A}_\lambda(0) = \mathcal{A}^*_\lambda(0) = F(f_\lambda)$ is connected. Let $Z(f_\lambda)$ denote the discrete set of zeros of f_λ and $P(f_\lambda)$ the set of poles of f_λ.

Consider the simple closed curve provided by Lemma 3.3. By the Argument Principle (see [9]),

$$\text{ind}(f_\lambda(\beta), 0) = -1 = \sum_{a \in Z(f_\lambda)} m(a)\, \text{ind}(\beta, a) - \sum_{a \in P(f_\lambda)} m(a)\, \text{ind}(\beta, a),$$

where $m(a)$ denotes the order of the zero or the pole.

Since β is a simple closed curve oriented counterclockwise, the equation reads

$$-1 = \sum_{a \in Z(f_\lambda)} m(a)\, \text{ind}(\beta, a) - \text{ind}(\beta, -1),$$

which can only be satisfied if β surrounds no zeros of f_λ and the unique pole $z = -1$ is surrounded by β.

So, $\beta \subset \mathcal{A}_\lambda(0) = \mathcal{A}^*_\lambda(0)$ and $-1 \in \text{int}(\beta)$. Then, the successive preimages of $\overline{\text{int}(\beta)}$ contain points $w \in \mathcal{O}^-_{f_\lambda}(\infty) \subset J(f_\lambda)$ which lie in the interior of a closed curve contained in $\mathcal{A}_\lambda(0)$. Hence, since the backward orbit of ∞ is an infinite set (the points that are eventually mapped to ∞ under iteration by f_λ), $\mathcal{A}_\lambda(0)$ is infinitely connected. □

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