REPORTS@SCM
AN ELECTRONIC JOURNAL OF THE
SOCIETAT CATALANA DE MATEMÀTIQUES

Bijective enumeration of constellations in higher genus

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Resum *(CAT)*
Bousquet-Mélou i Schaeffer donaren el 2000 una enumeració bijectiva de certs mapes anomenats constel·lacions. El 2019, Lepoutre va descriure una bijecció entre mapes biacoloribles de gènere arbitrari i certs mapes unicel·lulars del mateix gènere. Presentem una bijecció entre constel·lacions de gènere superior i certs mapes unicel·lulars que generalitza les dues bijeccions existents alhora.
Fent servir aquesta bijecció, enumerem una subclasse de constel·lacions sobre el tor, demostrant que llur funció generadora és una funció racional de la funció generadora de certs arbres.

Abstract *(ENG)*
Bousquet-Mélou and Schaeffer gave in 2000 a bijective enumeration of some planar maps called constellations. In 2019, Lepoutre described a bijection between bicolorable maps of arbitrary genus and some unicellular maps of the same genus. We present a bijection between constellations of higher genus and some unicellular maps that generalizes both existing bijections at the same time.
Using this bijection, we manage to enumerate a subclass of constellations on the torus, proving that its generating function is a rational function of the generating function of some trees.

Keywords: *combinatorics, maps, enumeration, bijection, blossoming map, constellation, rationality.*
MSC (2010): *05C10, 05C30.*
Received: *July 2, 2022.*
Accepted: *July 30, 2022.*

Acknowledgement
The author thanks Marie Albenque and Éric Fusy for their guidance during the elaboration of his bachelor's thesis, on which this paper is based. He also wishes to thank the anonymous referee for her/his useful comments.

Societat Catalana de Matemàtiques
Institut d'Estudis Catalans

1. Introduction

A *map* M of genus g is a proper embedding of a graph in $\mathcal{S}_g$, the torus with g holes, such that the maximal connected components of $\mathcal{S}_g \setminus M$ are contractible. These components are called *faces*. Multiple edges and loops are allowed. Maps are considered up to orientation preserving homeomorphisms. A *unicellular map* is a map with a single face.

Maps of genus 0 are called *planar maps*. They receive this name because embedding graphs in the sphere or in the plane is essentially the same. The stereographical projection, for instance, can produce a plane embedding from a sphere embedding. All the faces of a planar map embedded in the plane are contractible except for one, the *exterior face*, which is homeomorphic to the complement of a disk.

A *corner* of a map is a couple of consecutive edges around a vertex. Equivalently, a corner can be seen as an incidence between a face and a vertex. The *degree* of a vertex or face is its number of corners.

A *rooted map* is a map with a marked corner, which is called the *root corner* (or, simply, *root*). This root corner naturally defines a *root vertex* and a *root face*. The maps we consider here will always be rooted.

If a map is rooted, we have a notion of clockwise and counterclockwise when following contractible cycles. Precisely, we say that a tour around a contractible cycle is clockwise (resp. counterclockwise) if the root face lies on the left (resp. right) side of it.

In maps, edges join two (possibly equal) vertices and separate two (possibly equal) faces. Thus, given a map M, we can define its *dual map* M^* in the following way. The faces (resp. vertices) of M become the vertices (resp. faces) of M^* and the dual of an edge e joining vertices v_1 and v_2 and separating faces f_1 and f_2 is an edge e^* joining vertices f_1^* and f_2^* and separating faces v_1^* and v_2^*. Note that the dual of a corner is "itself" (i.e., the same vertex-face incidence) and that dualization is involutive: $(M^*)^* = M$.

Figure 1 contains an example of a planar map and its dual. Figure 2 contains an example of a map on the torus with some additional structure that is presented later. Maps on the torus are drawn on a square the parallel sides of which have to be identified.

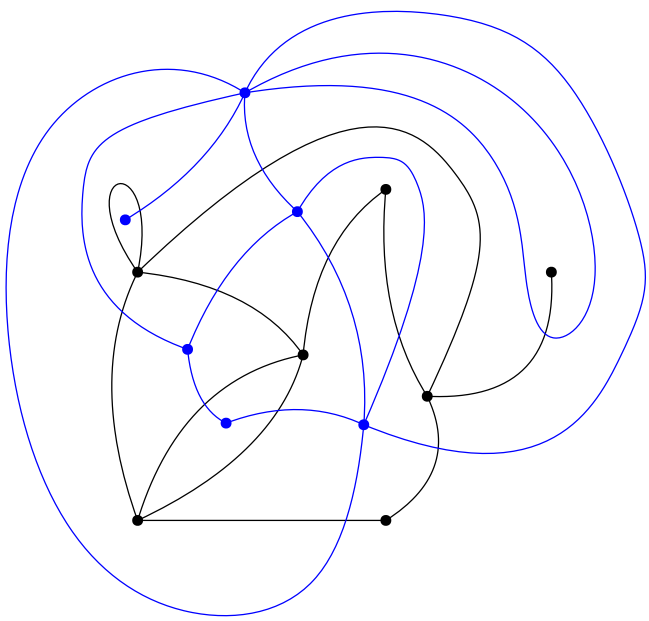


Figure 1: A planar map and its dual.

Maps are fundamental combinatorial objects that appear in many other fields of mathematics such as algebra and mathematical physics. The enumeration of planar maps began with the work of Tutte in the sixties [10]. In his work, Tutte enumerated a variety of families of maps, obtaining remarkably simple formulas. For example, he showed that the number of rooted planar maps with n edges is

$$\frac{2(2n)!3^n}{n!(n+2)!}.$$

His methods are based on the recursive combinatorial decomposition of maps and the equations obtained usually require the introduction of additional parameters called catalytic variables. In the late eighties, these techniques were extended to maps on surfaces of higher genus by Bender and Canfield [1, 2].

The simplicity of the formulas obtained by Tutte called for bijective demonstrations. Cori and Vauquelin gave the first bijective proof of the enumeration of planar maps in 1981 [7]. After them, many others continued this work, starting with Schaeffer, who gave numerous bijective constructions in the late nineties. In 1997, he introduced blossoming trees to formulate a new bijection for planar maps [9]. In 2000, Bousquet-Mélou and Schaeffer gave a bijection between planar constellations and some blossoming trees, which allowed them to prove enumerative formulas for constellations [3]. It should be mentioned that there is a second trend of bijections of maps based on trees decorated with some integers that encode metric properties of the maps. These bijections were applied to planar constellations in [4] and were later extended to higher genus in [5].

In positive genus, the natural equivalent of trees are unicellular maps. Chapuy, Marcus and Schaeffer introduced in [6] some techniques to analyse these unicellular maps by decomposing them into schemes with branches. In 2019, Lepoutre gave a blossoming bijection for bicolorable maps of any genus, which are a particular case of constellations [8].

Inspired by the work of Lepoutre, we reformulate the planar blossoming bijection of [3] in a way that naturally extends to higher genus. Thus, we obtain a blossoming bijection between constellations and some blossoming unicellular maps that also extends the bijection of [8]. Using this bijection, we are able to enumerate a particular case of constellations on the torus.

2. Constellations and m-bipartite unicellular maps

2.1 Constellations

Definition 2.1. Let $m \geq 2$. We say that a map whose faces are bicolored (black and white) is an *m-constellation* (Figure 2a) if

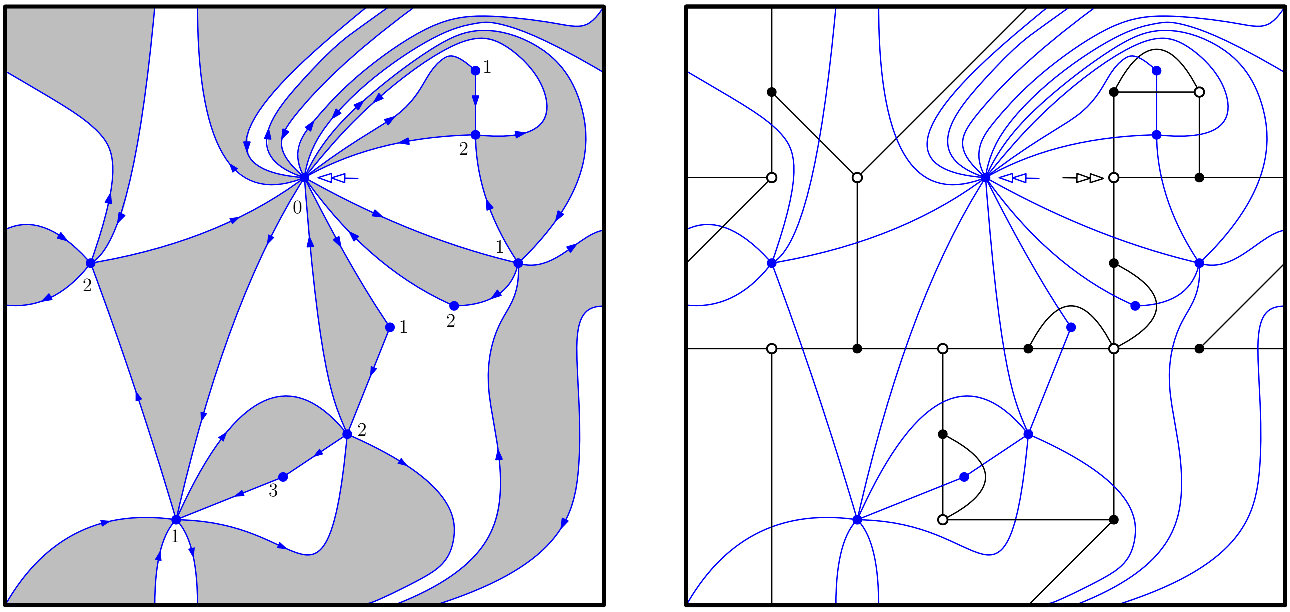
(i) adjacent faces have different colors,

(ii) black faces have degree m and white faces have degree mi for some integer $i \geq 1$ (which can be different among white faces),

(iii) vertices can be labeled with integers in $\{1, 2, \dots, m\}$ in such a way that turning clockwise around any black face the labels read $1, 2, \dots, m$.

A *rooted constellation* is a constellation that is rooted on a white corner. The first edge found when turning counterclockwise around the root vertex starting from the root corner is called the *root edge*. Note that the root corner can be recovered from the root edge, so it is equivalent to root a constellation on a white corner or on an edge.

The dual of an m-constellation is called an *m-Eulerian map* (Figure 2b). The dual of a rooted m-constellation (resp. rooted m-Eulerian map) is a rooted m-Eulerian map (resp. rooted m-constellation) with the "same" root. In other words, the root vertex, the root face and the root edge become, respectively, the root face, the root vertex and the root edge through dualization.



(a) A rooted 3-constellation of genus 1 endowed with its canonical orientation and labelling. The root is pointed by the double arrow.

(b) A rooted 3-constellation of genus 1 (blue) with its dual rooted 3-Eulerian map (black). Their roots are pointed by the double arrows.

Figure 2: A constellation and its dual map.

Consider a rooted m-constellation. The *canonical orientation* of its edges is the orientation for which its edges turn clockwise around black faces. When endowed with this orientation, the *canonical labelling* (Figure 2a) of its vertices is obtained by labelling every vertex with the length of the shortest oriented path to it from the root vertex.

This orientation and labelling was introduced by Bouttier, Di Francesco and Guitter in [4] for planar constellations to define what is now known as the BGD bijection.

2.2 Blossoming unicellular maps

Blossoming bijections were introduced by Schaeffer in [9] to put some classes of planar maps in bijection with decorated trees. These bijections consist in selecting a canonical spanning tree (or, more generally, a canonical spanning submap) and cut into two half-edges the edges not belonging to it. The resulting map is said to be a blossoming map, which can be *closed* back into the original map.

A *blossoming map* is a map with *stems* (that can be viewed as half-edges) attached to its vertices. There are two types of stems: *outstems*, which are outgoing stems, and *instems*, which are ingoing stems. Stems separate corners as if they were edges, which means that they count towards the (total) degree of their vertex. We will use the term *inner degree* when we want to ignore stems, i.e., when we only count the number of incident edges to a vertex (loops are counted twice).

A *rooted blossoming map* is a blossoming map with a marked instem, which is called the *root (instem)*. The vertex to which the root is attached is called the *root vertex* and the face incident to the root is called the *root face*. The corner on the right side of the root is called the *root corner*.

From now on we only consider blossoming maps that are unicellular.

The *good orientation* of a rooted unicellular blossoming map is the orientation for which every edge is, first, followed backwards and, then, forwards in tour around the unique face starting at the root corner. Note that it does not matter whether the face is followed clockwise or counterclockwise.

Given a rooted unicellular blossoming map which has m more instems than outstems, we can label its corners in the following way (Figure 3a). We make a counterclockwise tour around its unique face starting at the first corner after the root corner. Along this tour, we will visit every corner once and we will label it with the value of a counter that starts at $m-1$, increases by 1 every time we encounter an outstem and decreases by 1 every time we encounter an instem. The result of this procedure is called the *good labelling* of the unicellular blossoming map.

We say that an edge or stem *increases* (resp. *decreases*) by d if the value of its left label(s) minus the value of its right label(s) is d (resp. $-d$). Observe that, since there are m more instems than outstems, the last corner to label, which is the root corner, has good label 0.

2.3 m-bipartite unicellular maps

In [3], Bousquet-Mélou and Schaeffer define some objects called m-Eulerian trees and they construct a bijection between them and planar constellations. Here, we give a generalization of these objects to higher genus (m-bipartite unicellular maps) that we will show to be in bijection with constellations of higher genus.

Definition 2.2. Let $m \geq 2$. We say that a rooted unicellular blossoming map with m more instems than outstems and whose vertices are bicolored is an *m-bipartite unicellular map* (Figure 3a) if

(i) neighbouring vertices have different colors, instems are attached to white vertices and outstems are attached to black vertices,

(ii) black vertices have degree m,

(iii) white vertices have degree mi for some integer $i \geq 1$ (which can be different among white vertices),

and, when endowed with its good labelling,

(iv) the edges whose origin is a black vertex either decrease by 1 or increase by $m-1$,

(v) the edges whose origin is a white vertex decrease by $m-1$.

Given an m-bipartite unicellular blossoming map, consider the cyclic word formed by its stems in the order they appear in a counterclockwise tour around the face. Outstems are represented by the letter o and instems are represented by the letter i. Now we match letters o and i as if they were opening and closing parentheses, respectively. First, every letter o immediately followed by a letter i is *matched* with it. Then, all matched letters are removed and this procedure is repeated until no more matchings are possible (Figure 3b). Since there are exactly m more instems than outstems, m instems remain unmatched. We call these instems *single*. Note that the matching described is the only possible one, since, in a correct parenthesis word, an opening parentheses next to a closing one always have to be matched and can be ignored from that point on.

An m-bipartite unicellular map is *well-rooted* if its root instem is single. Well-rootedness can be characterized in the following way.

Proposition 2.3. *An m-bipartite unicellular map U is well-rooted if and only if its good labels are non negative.*

3. The bijection between m-constellations and m-bipartite unicellular maps

In this section we present our main result:

Theorem 3.1. *Rooted m-constellations of genus g with d_i white faces of degree mi are in bijection with well-rooted m-bipartite unicellular maps of genus g with d_i white vertices of degree mi.*

3.1 The closure Φ

We first describe how a well-rooted m-bipartite unicellular map can be closed to obtain an m-Eulerian map.

Definition 3.2. Let U be a well-rooted m-bipartite unicellular map. Let r be its root vertex. We define the *closure* $\Phi(U)$ of U in the following way (Figure 3).

First, every pair of matched stems b, l is connected to form a complete edge. The fact that the matched stems of U form a valid parentheses word ensures that these new edges can be drawn without intersections.

After this, there are m unmatched instems, including the root. Place a black vertex s with m outstems attached to it in the unique face and connect each of the outstems to a different unmatched instems. It is clear that this can also be done without intersections.

The final result, $\Phi(U)$, is a map. We choose to root it on the same corner as U or, equivalently, on the edge joining r and s.

Lemma 3.3. *The closure $\Phi(U)$ of a well-rooted m-bipartite unicellular map U of genus g with d_i white vertices of degree mi is a rooted m-Eulerian map of genus g with d_i white vertices of degree mi. Moreover, the good labelling of the corners of U corresponds to the canonical labelling of the faces of $\Phi(U)$.*

Proof sketch. The rules of good labels around stems ensure that the closure is a rooted m-Eulerian map. Moreover, the canonical labels are at least as large as the good ones because when turning clockwise around black vertices the good labels either increase by one or decrease by $m-1$, and the equality holds because there is a path from the root face to any other face that crosses only edges created by joining stems. □

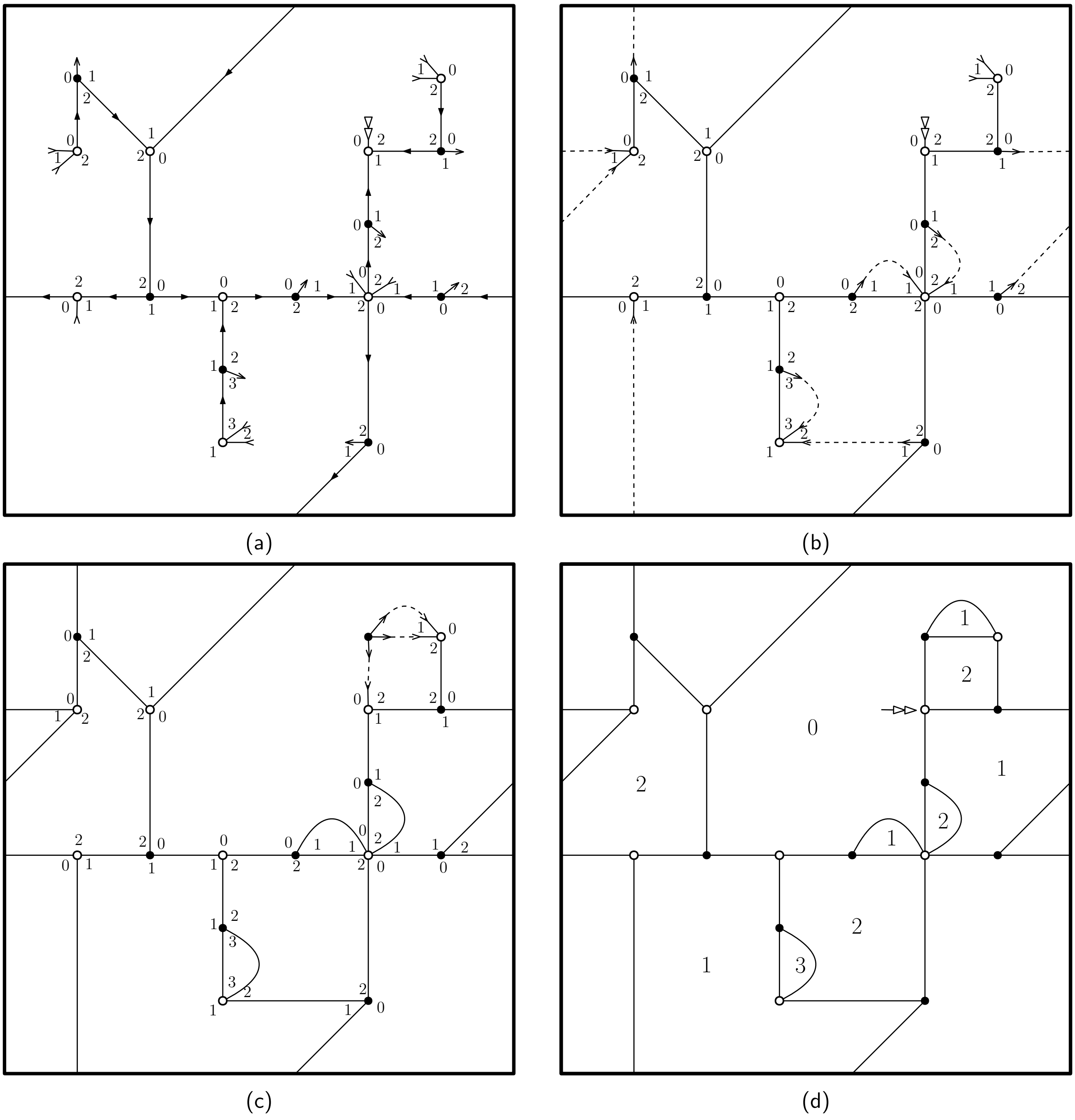


Figure 3: The closure of a well-rooted 3-bipartite unicellular map.

3.2 The opening Ψ

Here we do the inverse transformation, that is, starting from a rooted m-Eulerian map, cut some of its edges into stems so that the result is a well-rooted m-bipartite unicellular map.

Definition 3.4. Let M be a rooted m-Eulerian map. We define its *opening* $\Psi(M)$ in the following way (Figure 4). First, consider the dual map C of M, which is a rooted m-constellation. Endow C with its canonical orientation and labelling and take its leftmost Breadth-First Search (BFS) exploration tree T. For every edge of M whose dual belongs to T, cut it into two stems: an instem attached to the white vertex and an outstem attached to the black one. Finally, cut the root edge and remove s.

We root the result of this, $\Psi(M)$, at the instem created when cutting the root edge.

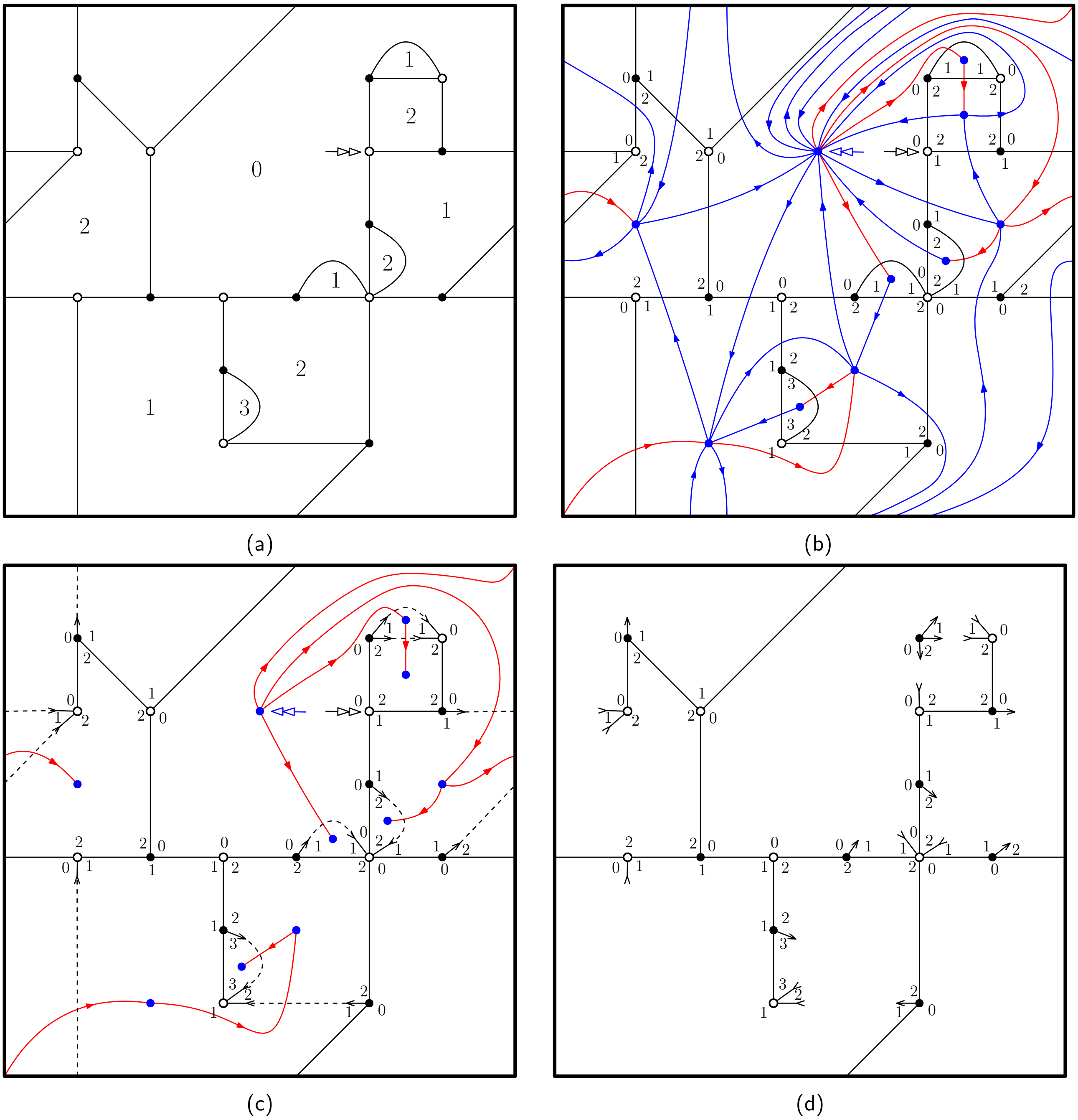


Figure 4: The opening of a rooted 3-Eulerian map.

Lemma 3.5. *The opening* $\Psi(M)$ *of a rooted* m*-Eulerian map* M *of genus* g *with* d_i *white vertices of degree* mi *is a well-rooted* m*-bipartite unicellular map of genus* g *with* d_i *white vertices of degree* mi*. Moreover, the canonical labelling of* M *corresponds to the good labelling of* $\Psi(M)$ *and*

$$\Phi(\Psi(M)) = M.$$

Proof sketch. The only difficulty here is to show that, after the opening, there can be no edge oriented from white to black that increases by 1. If there was one such edge, the leftmost BFS tree would have seen, first, its left side and, then, its right side, which would be a contradiction. □

So far we have shown that the closure of a well-rooted m-bipartite unicellular map is a rooted m-Eulerian map and that the opening of a rooted m-Eulerian map is a well-rooted m-bipartite unicellular map whose closure is the original map. To show that Φ and Ψ are inverse operations and, thus, to prove Theorem 3.1, we just need the following lemma.

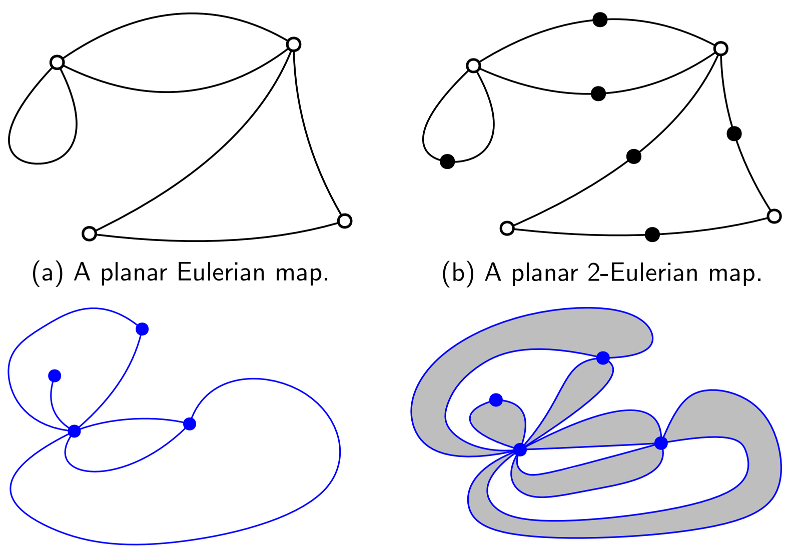
Lemma 3.6. *Let* U *be a well-rooted* m*-bipartite unicellular map. Then,*

$$\Psi(\Phi(U)) = U.$$

Proof sketch. Similarly, here one needs to prove that the duals of the edges that are created during the closure by joining stems form a leftmost BFS tree. □

Remark 3.7. The m-Eulerian trees described in [3] by Bousquet-Mélou and Schaeffer are the planar instances of the m-bipartite unicellular trees we have introduced here. We use the same closing operation as they do, but flipping the orientation of the surface, which amounts to swapping the notions of left-right and clockwise-counterclockwise. Thus, when we restrict our bijection to the sphere, we recover their bijection.

Remark 3.8. In [8], Lepoutre gives a bijection between bicolorable maps of arbitrary genus and an adequate family of blossoming unicellular maps. It is easy to convince oneself that bicolorable maps are, in fact, 2-Eulerian maps whose black vertices have been replaced by a single edge connecting their two white neighbours (Figure 5).



(a) A planar Eulerian map.

(b) A planar 2-Eulerian map.

(c) The dual map of 5a.

(d) The dual map of 5b: a planar 2-constellation.

Figure 5: The relation between bicolorable maps and 2-Eulerian maps.

In his bijection, Lepoutre opens bicolorable maps in a way that he shows to be equivalent to the following one. First, take the dual of the bicolorable map, which is a bipartite map, and endow it with its geodesic orientation. Then, consider the leftmost BFS exploration tree of this oriented bipartite map and, finally, cut all the edges of the bicolorable map whose dual does not belong to the tree. This is essentially the same way in which we open 2-Eulerian maps: the difference between our canonical orientation of 2-constellations (we orient vertices clockwise around black faces) and Lepoutre's geodesic orientation of the bipartite map is explained by the fact that he has collapsed the black faces of the 2-constellation into edges to obtain the bipartite map. Therefore, we can say that our bijection also generalizes the one given by Lepoutre in [8].

4. Rerooting an m-bipartite unicellular map

After Theorem 3.1 has been established, one can try to enumerate rooted constellations by enumerating well-rooted m-bipartite unicellular maps.

The first problem we run into when trying to count well-rooted m-bipartite unicellular maps is precisely the fact that they are well-rooted. As Lepoutre explains in [8], well-rootedness is a global notion, since it requires the positivity of all the good labels of a map. This complicates the task of counting these objects and, thus, we would like to get rid of it. In order to do so, we use the technique of rerooting first introduced in [9] and which was successfully used in [3] and [8]. Specifically, we provide an algorithm to reroot a well-rooted map on any instem, which will later yield an enumerative relation between m-bipartite unicellular maps and well-rooted m-bipartite unicellular maps.

Definition 4.1. Let U be a rooted m-bipartite unicellular map, let r be its root and let t be a distinguished instem of U. We endow U with its good orientation and its good labelling.

The *rerooting* algorithm is defined as follows. If $t = r$, we do nothing. Otherwise, we first join r and t to create an edge. This divides the single face of U into faces f_L and f_R, where f_L is the one containing the root corner of U. We then add m to all labels of f_L and we reverse the orientation of all the edges that separate f_L and f_R. Finally, we cut the edge joining r and t back into two instems and we swap the roles of r and t: t becomes the root and r becomes the distinguished instem.

The rerooting procedure always produces a valid m-bipartite unicellular map. This is why we say that these maps are stable under rerooting. Furthermore, it allows us to prove the following:

Proposition 4.2. *m-bipartite unicellular maps with a distinguished single instem are in bijection with well-rooted m-bipartite unicellular maps with a distinguished instem.*

5. Enumeration of bipartite 3-face-colorable cubic maps on the torus

In this section, we prove our second theorem:

Theorem 5.1. *Bipartite* 3*-face-colorable cubic maps of genus* 1 *are enumerated by*

$$C(z) = \frac{T(z)^3}{(1 - T(z))(1 - 4T(z))^2},$$

where z marks the number of white vertices and $T(z)$ is the unique generating function satisfying $T(z) = z + 2T(z)^2$. In particular, $C(z)$ is a rational function of $T(z)$.

Bipartite 3-face-colorable cubic maps of genus 1 are 3-Eulerian maps of genus 1 whose white vertices all have degree 3. This is a very particular case compared to the general m-constellations of arbitrary genus for which we have built a bijection, but it allows for relatively simple calculations that can be done by hand.

Generating functions are formal power series whose n-th coefficient equals the number of objects of size n in some combinatorial class. For example, if $\mathcal{C}$ is the class of rooted bipartite 3-face-colorable cubic maps of genus 1, counted by their number of white vertices, then $C(z) = \sum_{n\geq 0} c_n z^n$ is their generating function in the sense that there are exactly c_n such maps with n white vertices. We use the *Symbolic Method* to translate the relations between the combinatorial classes (classes of graphs in our setting) into equations involving their generating functions.

Let $\mathcal{O}$ be the class of well-rooted 3-bipartite unicellular maps of genus 1 whose white vertices have degree 3, counted by their number of instems. Since the number of instems of a map $o \in \mathcal{O}$ is equal to the number of white vertices of its closure $c \in \mathcal{C}$, $C(z) = O(z)$.

Let $\mathcal{U}$ be the class of 3-bipartite unicellular maps of genus 1 whose white vertices have degree 3 counted by their number of instems different from the root. By Proposition 4.2, we have the following.

Lemma 5.2. *The generating functions of $\mathcal{O}$ and $\mathcal{U}$ satisfy the relation*

$$O(t) = 3\int_0^t U(z)\,dz.$$

5.1 The pruned maps and their enumeration

We follow the framework introduced by Chapuy, Marcus and Schaeffer in [6] to study unicellular maps.

The *extended scheme* of a map $u \in \mathcal{U}$ is the map obtained by, first, removing all its stems and, then, iteratively removing all its vertices of degree 1. This procedure only removes stems and treelike parts from the map, so an extended map is also a unicellular map. In fact, any map $u \in \mathcal{U}$ can be decomposed into an extended scheme and some attached stems and treelike parts.

An extended scheme can only have vertices of degree 2, which we call *branch vertices*, and vertices of degree 3, which we call *scheme vertices*. The treelike parts can only be attached to white branch vertices. In our setting, there are always exactly two scheme vertices and they are black.

Let $\mathcal{T}$ be the class of these attachable treelike parts, counted by their number of instems. For the sake of simplicity, we will consider that a single instem is a treelike part and belongs to $\mathcal{T}$. It is easy to see that the generating function of $\mathcal{T}$ satisfies the following recursive relation:

$$T(z) = z + 2T(z)^2.$$

Let $u \in \mathcal{U}$ be a 3-bipartite unicellular map whose white vertices have degree 3. Its *pruned map* p is obtained by replacing all its treelike parts by instems. The treelike part containing the root is replaced by a root instem. Let $\mathcal{P}$ be the class obtained by pruning every map in $\mathcal{U}$. The pruned maps of $\mathcal{P}$ are counted by their number of instems different from the root. Observe that, if we keep the good labels on the pruned map, the rules of the labelling still apply. In other words, $\mathcal{P} \subset \mathcal{U}$.

Lemma 5.3. *The generating functions of $\mathcal{U}$, $\mathcal{P}$ and $\mathcal{T}$ satisfy the relation*

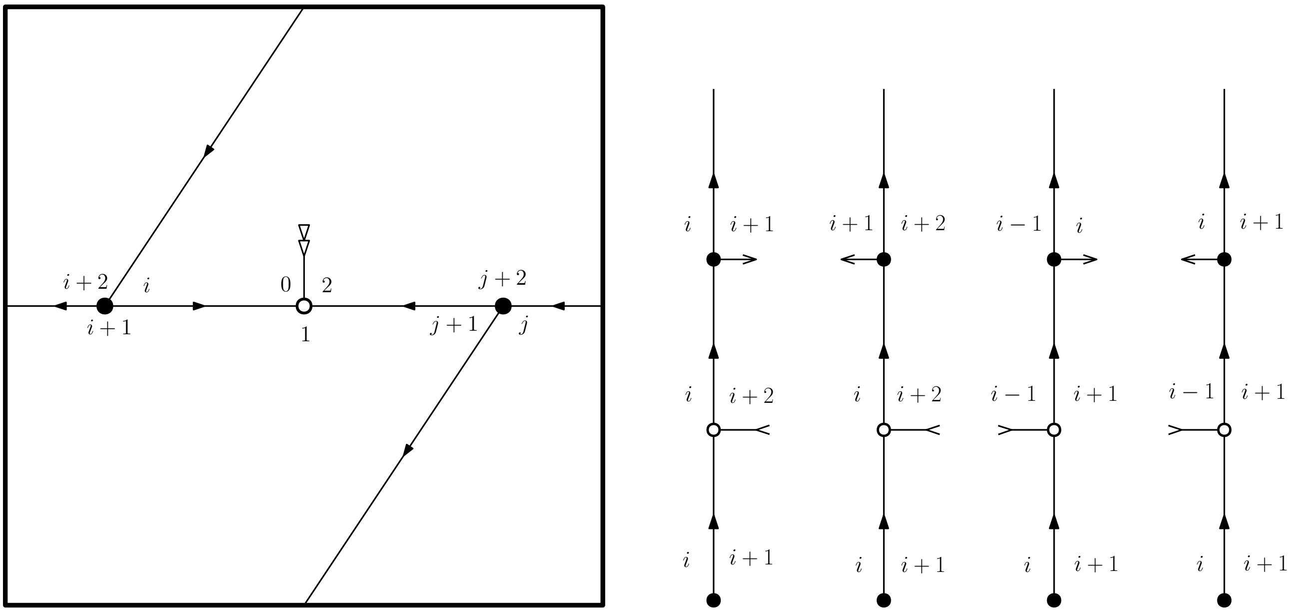
$$U(z) = \frac{\partial T}{\partial z} P(T(z)).$$

Proof sketch. Each stem in the pruned map is replaced by a tree and the root stem is replaced by a tree with a marked leaf that becomes the new root. □

We would now like to enumerate $\mathcal{P}$.

The *labelled scheme* of a pruned map $p \in \mathcal{P}$ is obtained by removing all its branch vertices except for the one where the treelike part containing the root was attached. The good labels of the remaining corners are kept.

It is clear that labelled schemes are uniquely determined by the lowest label on each of its scheme vertices (Figure 6a). There is, thus, a correspondance between labelled schemes and pairs $(i, j) \in \mathbb{Z}^2$.



(a) A generic labelled scheme.

(b) The first step of a branch.

Figure 6: Counting the pruned maps.

The labelled scheme associated to the pair (i, j) will be denoted $l_{i,j}$, and the subclass of pruned maps that have $l_{i,j}$ as labelled scheme will be denoted $\mathcal{P}_{i,j}$. Given $(i, j) \in \mathbb{Z}^2$, we want to compute $P_{i,j}(z)$. To do so, we replace every edge of $l_{i,j}$ by a valid branch whose labels agree with the labels of $l_{i,j}$. A branch starts at a black vertex. There are four ways to place the stems of the first two vertices (Figure 6b).

The generating functions of branches are obtained by using *weighted Motzkin paths*. Multiplying the four branches in a given $l_{i,j}$ and summing over all pairs (i, j) gives the following:

$$P = \sum_{i,j \in \mathbb{Z}} P_{i,j} = \cdots = \frac{z^2(2z-1)}{(z-1)^2(4z-1)^3}.$$

We can finally conclude the proof of Theorem 5.1:

$$\begin{aligned} C(z) = O(t) &= 3\int_0^t U(z)\,dz = 3\int_0^t \frac{\partial T}{\partial z} P(T(z))\,dz \\ &= 3\left(\int_0^z P(\eta)\,d\eta\right)_{|z=T(t)} = \frac{T(z)^3}{(1-T(z))(1-4T(z))^2}. \end{aligned}$$

In view of Theorem 5.1, we can formulate the following conjecture.

Conjecture 5.4. *Bipartite* 3*-face-colorable cubic maps of arbitrary genus are enumerated by a generating function which is a rational function of* $T(z)$*.*

Since blossoming bijections in [8] produce enumerative results in which there is scheme by scheme rationality, we hope that will also be the case for these maps.

References

[1] E.A. Bender, E.R. Canfield, "The asymptotic number of rooted maps on a surface", *J. Combin. Theory Ser. A* **43**(2) (1986), 244–257.

[2] E.A. Bender, E.R. Canfield, "The number of rooted maps on an orientable surface", *J. Combin. Theory Ser. B* **53**(2) (1991), 293–299.

[3] M. Bousquet-Mélou, G. Schaeffer, "Enumeration of planar constellations", *Adv. in Appl. Math.* **24**(4) (2000), 337–368.

[4] J. Bouttier, P. Di Francesco, E. Guitter, "Planar maps as labeled mobiles", *Electron. J. Combin.* **11**(1) (2004), Research Paper 69, 27 pp.

[5] G. Chapuy, "Asymptotic enumeration of constellations and related families of maps on orientable surfaces", *Combin. Probab. Comput.* **18**(4) (2009), 477–516.

[6] G. Chapuy, M. Marcus, G. Schaeffer, "A bijection for rooted maps on orientable surfaces", *SIAM J. Discrete Math.* **23**(3) (2009), 1587–1611.

[7] R. Cori, B. Vauquelin, "Planar maps are well labeled trees", *Canadian J. Math.* **33**(5) (1981), 1023–1042.

[8] M. Lepoutre, "Blossoming bijection for higher-genus maps", *J. Combin. Theory Ser. A* **165** (2019), 187–224.

[9] G. Schaeffer, "Bijective census and random generation of Eulerian planar maps with prescribed vertex degrees", *Electron. J. Combin.* **4**(1) (1997), Research Paper 20, 14 pp.

[10] W.T. Tutte, "A census of planar maps", *Canadian J. Math.* **15** (1963), 249–271.