REPORTS@SCM

AN ELECTRONIC JOURNAL OF THE
SOCIETAT CATALANA DE MATEMÀTIQUES

On the support of Zygmund measures

*Laia Weisz Font

Universitat Autònoma de Barcelona (UAB)

laia.weisz@gmail.com

*Corresponding author

Resum *(CAT)*

Hem estudiat els compactes que suporten mesures de Zygmund, dels quals no se'n coneix cap caracterització. Hem introduït un concepte anomenat log-porositat, que proporciona una condició suficient per tal que un compacte no pugui ser el suport d'una mesura de Zygmund. Hem vist que aquest resultat no deriva dels treballs de Makarov ni Kaufman. Hem introduït el concepte de capacitat Zygmund d'un compacte i hem proposat una caracterització dels suports de les mesures de Zygmund en termes d'aquesta capacitat. Hem demostrat que els compactes pels quals el límit d'aquesta capacitat és zero, no poden suportar mesures de Zygmund.

Abstract *(ENG)*

We analyse the compact sets that are the support of Zygmund measures, of which no characterisation is known. We introduce the concept of log-porosity which provides a sufficient condition that guarantees that a compact cannot be the support of a Zygmund measure. This result does not derive from the results of Makarov and Kaufman. We introduce the concept of Zygmund capacity of a compact and propose a characterisation of the supports of Zygmund measures in terms of this capacity. We prove that the compact sets for which the limit of this capacity is zero cannot be the support of Zygmund measures.

Keywords: *Zygmund measures, Kaufman's Theorem, log-porosity.*

MSC (2010): *28A12, 28A05, 03E15.*

Received: *March 27, 2022.*

Accepted: *July 27, 2022.*

Acknowledgement

I would like to thank Juan J. Donaire for his invaluable guidance and support.

Societat Catalana de Matemàtiques

Institut d'Estudis Catalans

1. Introduction

A positive finite measure μ in $\mathbb{R}$ is called a Zygmund measure if there exists a constant C such that $|\mu(I) - \mu(I')| \le C\,|I|$ for all pairs of adjacent intervals I, I' of the same length $|I|$. The infimum of the values of C for which the inequality holds is called the Zygmund norm of μ and is denoted by $\|\mu\|_*$.

It can be seen that a positive measure is a Zygmund measure if and only if its distribution function belongs to the Zygmund space Λ_*, defined as the set of real-valued and bounded functions such that

$$\|f\|_* = \sup_{x,h\in\mathbb{R}^n} \frac{|f(x+h)+f(x-h)-2f(x)|}{\|h\|} < +\infty. \tag{1}$$

Let us recall the definition of Hausdorff measure. A measure function is an increasing continuous function $\varphi\colon [0,\delta) \to \mathbb{R}^+$ such that $\varphi(0) = 0$. Let $E \subset \mathbb{R}$ be a bounded set, we define the Hausdorff measure of E with respect to φ as

$$H_\varphi(E) = \lim_{\varepsilon\to 0} \inf \left\{ \sum_j \varphi(|I_j|) : E \subset \bigcup_j I_j \quad \text{and} \quad |I_j| \le \varepsilon \right\}.$$

If $\varphi(t) = t^\alpha$ for some $\alpha > 0$, then we will write $H_\alpha(E)$.

Frostman's Lemma ([7, p. 112]) states that a compact set K is the support of a Lip_α measure if and only if $H_\alpha(K) > 0$. As a consequence, compact sets that are the support of Lip_α measures are completely determined. Although Zygmund measures can be considered as the limit of Lip_α measures as $\alpha \to 1$ (see [8]), a characterisation of the compact sets that are the support of Zygmund measures is not known.

2. Preliminary results

Clearly, the Lebesgue measure restricted to a positive measure set is a Zygmund measure. The following theorem proves the existence of a Zygmund measure whose support has zero Lebesgue measure.

Theorem 2.1 (Kahane)**.** *There exists a positive singular Zygmund measure.*

Sketch of proof. In [3], Kahane proved his theorem by geometrically building a Zygmund measure whose support has zero Lebesgue measure.

Let Q_n denote the tetradic intervals of length 4^{-n} and let us define the following succession of simple functions: $s_0 \equiv 1$ and $s_n(x) = s_{n-1}(x) + \varepsilon_n(x)$, where

$$\varepsilon_n(x) = \begin{cases} -1 & \text{if } x \in I_1 \cup I_4, \\ 1 & \text{if } x \in I_2 \cup I_3, \end{cases}$$

and $I_1, I_2, I_3, I_4 \in Q_n$ such that[1] $I_1 \cup I_2 \cup I_3 \cup I_4 = I_j \in Q_{n-1}$ and $x \in I_j$.

[1] We consider I_k to the left of I_{k+1}, for $k = 1, 2, 3$.

Let us consider the stopping time $\tau(x) = \inf\{n : s_n(x) = 0\}$. Cleary, $\tau(x) < \infty$ for almost all x. Let us build the succession of measures defined by $\mathrm{d}\mu_n = s_{n\wedge\tau}\,\mathrm{d}x$, where $n \wedge \tau = \min\{n, \tau\}$ and $\mathrm{d}x$ is the Lebesgue measure. It can be seen that μ_n is a positive probability measure. In addition, if we denote by μ the limit of μ_n as $n \to \infty$, it can be seen that μ is a Zygmund measure and its support has zero Lebesgue measure. □

Theorem 2.2 (Makarov)**.** *If μ is a positive Zygmund measure, then μ is absolutely continuous with respect to H_Φ, where*

$$\Phi(t) = t\sqrt{\log\left(\frac{1}{t}\right)\cdot\log\log\log\left(\frac{1}{t}\right)}. \tag{2}$$

In addition, there exists a Zygmund measure ν which satisfies $0 < H_\Phi(\mathrm{supp}\,(\nu)) < \infty$.

Makarov's Theorem provides the optimal measure function ϕ such that if a compact set K has $H_\phi(K) = 0$, then it cannot be the support of a Zygmund measure. The proof of the first and second statements of Makarov's Theorem can be found in [6] and [5] respectively.

Theorem 2.3 (Kaufman)**.** *For each measure function h such that $\lim_{t\to 0}\frac{t}{h(t)} = 0$, there exists a compact set K with $H_h(K) > 0$ which is not the support of a Zygmund measure.*

Kaufman's Theorem implies that it is impossible to characterise the supports of Zygmund measures in terms of Hausdorff measures. In [4], Kaufman proved his theorem by introducing a special class of sets: A compact set E is called *porous* (with a parameter $a > 0$) if, for each $\delta > 0$, there exists a covering of E by disjoint open intervals $I_{\varepsilon_t}(x_t) = (x_t - \varepsilon_t, x_t + \varepsilon_t)$ such that $\varepsilon_t < \delta$ and each interval $I_{\varepsilon_t}(x_t)$ contains an interval $I_{a\varepsilon_t}(x'_t) = (x'_t - a\varepsilon_t, x'_t + a\varepsilon_t) \subset I_{\varepsilon_t}(x_t)$ disjoint from E.

Kaufman proved that porous sets cannot be the support of Zygmund measures and that for each function h that satisfies the hypothesis of Theorem 2.3, there exists a porous set K with $H_h(K) > 0$.

Proof. Let $S = \{n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots\}$ be a sequence of positive integers whose complement is infinite and let us define

$$E = \left\{\sum_{k=1}^{\infty}\varepsilon_k\,2^{-n_k}\ \middle|\ \varepsilon_k \in \{0,1\}\quad \forall k\right\}.$$

Let m be an integer not in S and let $x \in E$. Then $a \le 2^{m-1}x \le a + \frac{1}{2}$ for some $a \in \mathbb{N}_0$ and consequently, each element of E has distance $\le d = 2^{-m-1}$ from one of the centres $\{(q+\frac{1}{4})2^{1-m} \mid q \in \mathbb{N}_0\}$. The distance between two consecutive centres is $4d$, hence E is porous.

We need to define S so that $H_h(E) > 0$. Let ψ be a positive function such that $\lim_{t\to 0^+} t\cdot\psi(t) = 0$ and $\lim_{t\to 0^+} h(t)\cdot\psi(t) = \infty$. We build S as $\{\lfloor -\log_2(\psi^{-1}(2^k))\rfloor : k \in \mathbb{N}\}$. It can be seen that, with this definition, $|S^c \cap \mathbb{N}| = \infty$ and $\lim_{k\to\infty} 2^k h(2^{-n_k}) = \infty$. Let ν be a probability measure with support in E such that each interval $I \in \mathfrak{D}_{n_k}$ has measure $\mathcal{O}(2^{-k})$. Let I be an interval of length r small and let k be an integer such that $2^{-n_k} \ge r > 2^{-n_{k+1}}$. We define J as the interval of length 2^{-n_k} and the same centre as I. Therefore,

$$\nu(I) \le \nu(J) \le \mathcal{O}(1)\cdot\nu(I_j^{n_{k+1}}) = \mathcal{O}(1)\cdot 2^{-k-1} < \mathcal{O}(1)\,h(2^{-n_{k+1}}) \le \mathcal{O}(1)\,h(r) = \mathcal{O}(1)\,h(|I|),$$

as $\lim_{k\to\infty} 2^k h(2^{-n_k}) = \infty$. Consequently, $H_h(E) = \lim_{\varepsilon\to 0}\inf\{\sum_j h(|I_j|) : E \subset \bigcup I_j,\ |I_j| \le \varepsilon\} > 0$.

We will introduce a generalisation of porosity in order to prove that porous sets cannot be the support of a nontrivial Zygmund measure. □

Given a compact set $K \subset \mathbb{R}$ and a closed interval I, let us denote by I^* the biggest open interval in I, disjoint from K and satisfying $2|I^*| \le |I|$.

Definition 2.4. Let $K \subset \mathbb{R}$ be a compact set of zero Lebesgue measure. We say K is *log-porous* if

$$\liminf_{\varepsilon \to 0} \left\{ \sum_{j\ge 1} |I_j| \log\left(\frac{|I_j|}{|I_j^*|}\right) : K \subset \bigcup_{j \ge 1} I_j, \text{ with pairwise disjoint interiors and } |I_j| < \varepsilon \right\} = 0.$$

Theorem 2.5. *A log-porous set cannot be the support of a Zygmund measure.*

Theorem 2.6. *There exists a log-porous compact set K such that it is nonporous and $H_\Phi(K) > 0$, where Φ is the function defined in (2).*

It can be easily seen that porous sets are log-porous. Theorem 2.6 implies that Theorem 2.5 is not a consequence of Makarov's Theorem or Kaufman's Theorem. In order to prove Theorem 2.5 we will need the following result.

Proposition 2.7. *Let $f \in \Lambda_*$. Then for any $t \in (0,1)$ and $a, b \in \mathbb{R}$,*

$$|(1-t)f(a) + tf(b) - f((1-t)a + tb)| \le C\|f\|_* \varphi(t)|b-a|,$$

where C is an absolute constant and $\varphi(t) = t\log\frac{1}{t}$ if $t \le 1/2$ and $\varphi(t) = \varphi(1-t)$ if $t \ge 1/2$.

The proof of Proposition 2.7 can be found in [1].

Proof of Theorem 2.5. Let $K \subset (0,1)$ be a log-porous compact set. Let μ be a positive Zygmund measure with support in K and let f be its distribution function. Given $\eta > 0$ small, there exists $\varepsilon > 0$ and a covering by closed intervals $\{I_j\}$ given by the definition of log-porosity, such that $|I_j| < \varepsilon$ $\forall j \ge 1$ and $\sum_{j\ge1} |I_j| \log(\frac{|I_j|}{|I_j^*|}) < \eta$.

Let us denote $I_j = (a_j, b_j)$ and $I_j^* = (c_j, d_j) \subset I_j$. We define $\rho_j = \frac{|I_j^*|}{|I_j|}$ and $x = \frac{c_j - \rho_j a_j}{1-\rho_j} = \frac{d_j - \rho_j b_j}{1-\rho_j}$. Note that $f(d_j) = f(c_j)$ since $K \cap (c_j, d_j) = \emptyset$. By Proposition 2.7, we have

$$\begin{aligned}
\rho_j|f(b_j) - f(a_j)| &= |\rho_j f(b_j) - \rho_j f(a_j) + f(c_j) - f(d_j) + (1-\rho_j)f(x) - (1-\rho_j)f(x)| \\
&\le |\rho_j f(b_j) + (1-\rho_j)f(x) - f(d_j)| + |\rho_j f(a_j) + (1-\rho_j)f(x) - f(c_j)| \\
&\le C\varphi(\rho_j)(b_j - x) + C\varphi(\rho_j)(a_j - x) = C\varphi(\rho_j)(b_j - a_j),
\end{aligned}$$

and, as a consequence, $|f(b_j) - f(a_j)| \le C|b_j - a_j|\frac{1}{\rho_j}\varphi(\rho_j) = C|b_j - a_j|\log(\frac{1}{\rho_j}) = C|I_j|\log(\frac{|I_j|}{|I_j^*|})$, since $\rho \le \frac{1}{2}$. Therefore $\mu(K) = f(1) - f(0) \le C\sum_{j\ge1}|I_j|\log(\frac{|I_j|}{|I_j^*|}) < C\cdot\eta$.

As η is arbitrarily small, we conclude that $\mu(K) = 0$ and μ must be the trivial measure. □

Proof of Theorem 2.6. Let us consider, in $[0,1]$, the function defined by

$$\varphi(t)=\begin{cases}0 & \text{if } t=0,\\ \dfrac{t}{\sqrt[4]{\log_2(2/t)}} & \text{if } t\in(0,1].\end{cases}$$

Note that φ is increasing and convex, therefore $2\varphi(2^{-n-1})<\varphi(2^{-n})$.

The compact will be constructed inductively in a similar way as the Cantor set. Let $E_0=I_1^0=[0,1]$ and let $E_n=\bigcup_{j=1}^{2^n}I_j^n$. For each closed interval I_j^n, we consider I_{2j-1}^{n+1} and I_{2j}^{n+1} the two closed corner intervals of I_j^n of length $\varphi(2^{-n-1})$. Then we construct E_{n+1} as $E_{n+1}=\bigcup_{j=1}^{2^{n+1}}I_j^{n+1}$. Finally, we set $K=\bigcap_{n\geq 1}E_n$.

Firstly, we will see that K is log-porous. For $n\geq 0$, let us consider the covering of K given by $\bigcup_{j=1}^{2^n}I_j^n$. By construction, for each I in the covering, the length of I^* is $\varphi(2^{-n})-2\varphi(2^{-n-1})$. As a result,

$$\sum_{j=1}^{2^n}|I_j^n|\log\left(\frac{|I_j^n|}{|I_j^{n*}|}\right)=2^n\varphi(2^{-n})\log\left(\frac{\varphi(2^{-n})}{\varphi(2^{-n})-2\varphi(2^{-n-1})}\right)=\frac{1}{\sqrt[4]{n+1}}\log\left(\frac{\sqrt[4]{n+2}}{\sqrt[4]{n+2}-\sqrt[4]{n+1}}\right)$$

which goes to 0 as n tends to infinity. Now we will see that $H_\Phi(K)>0$. Let us define $\lambda_0=1$ and $\lambda_n=\frac{1}{2}\lambda_{n-1}(x)$ if $x\in I_j^n$ for some j and $\lambda_n=0$ otherwise. We consider the succession of measures $d\nu_n=\varphi(2^{-n})\lambda_n\,dx$, and we denote by ν the limit of ν_n. Note that ν is a positive probability measure with support in K and such that $\nu(I_j^n)=2^{-n}$ for all j.

By a similar argument as the one used in the proof of Theorem 2.3, we conclude that $H_{\varphi^{-1}}(K)>0$. Since $\varphi^{-1}(t)=o(\Phi(t))$ as $t\to 0^+$, by the comparison lemma between Hausdorff measures (see [2, p. 60]), we conclude that $H_\Phi(K)=\infty$.

In order to prove that K is nonporous, we assign, to each $x\in K$, a succession $(\delta_n(x))$ of $0's$ and $1's$ in the following way: if $x\in I_{2j-1}^n$ for some j, we set $\delta_n(x)=0$; if $x\in I_{2j}^n$, we set $\delta_n(x)=1$. Let E be the set of $x\in K$ such that there exists $n_0=n_0(x)$ such that, for $n\geq n_0$, $\delta_n(x)\neq\delta_{n+1}(x)$.

We assume that K is porous with parameter $0<\rho<1$ and consider the associated covering of K. If $x\in E$, let I be the interval of the aforementioned covering containing x. We choose $n\in\mathbb{N}$ such that $\varphi(2^{-n-1})<|I|\leq\varphi(2^{-n})$. By election of x, the length of the biggest interval contained in I and disjoint of K is less than $\varphi(2^{-n+2})-2\varphi(2^{-n+1})$. Therefore, $\rho\varphi(2^{-n-1})\leq\rho\cdot|I|\leq\varphi(2^{-n+2})-2\varphi(2^{-n+1})$ and, as a result

$$\frac{\varphi(2^{-n+2})-2\varphi(2^{-n+1})}{\varphi(2^{-n-1})}\geq\rho.$$

We have reached a contradiction, since the left expression goes to 0 as n tends to infinity. □

3. An approach to the characterisation of the supports of Zygmund measures

Our aim is to find a characterisation of the compact sets that support Zygmund measures, *i.e.*, the compact sets K for which $\sup\{\mu(K):\operatorname{supp}(\mu)\subset K,\ \mu\geq 0,\ \|\mu\|_*\leq 1\}>0$, where μ is a Zygmund measure.

To that end, we will approximate K as the union of closed dyadic intervals and we will determine the maximum mass a fixed-norm Zygmund measure can have. We will use the following notation to denote the dyadic intervals of length 2^{-n}:

$$\mathfrak{D}_n = \left\{ \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right) \;\middle|\; k \in \{0, 1, \dots, 2^n - 1\} \right\}.$$

Lemma 3.1. *Let $K \subset (0,1)$ be the union of 2^{-n}-length closed dyadic intervals. Let μ be a positive measure with $\operatorname{supp} \mu \subseteq K$ and constant density over each interval in $\mathfrak{D}_n$. If the following condition holds, μ is a Zygmund measure with support in K and $\|\mu\|_* \sim C$.*

$$|\mu(I) - \mu(I')| \leq C|I|, \text{ where } I,\ I' \in \mathfrak{D}_k \text{ are adjacent and } k \leq n.$$

This lemma can be easily proven using a variation of the proof of Kahane's Theorem and we will use it to attempt to determine a geometrical characterisation of the compact sets that support a Zygmund measure. With that goal, we shall introduce the concept of $\mathcal{Z}$-2^k sequences.

Definition 3.2. Let $n \in \mathbb{N}$ and $0 \leq k \leq n$. A number sequence $x_1 x_2 \dots x_{2^{n-k}}$ is said to be $\mathcal{Z}$-2^k if the following conditions hold.

$$\begin{cases} x_j \geq 0 & \forall j, \\ |x_j - x_{j-1}| \leq 2^k & j = 2, \dots, 2^{n-k}, \\ x_j \leq 2^k & j = 1,\ 2^{n-k}. \end{cases}$$

Let $K \subseteq [0,1]$ be a compact set and let $n \in \mathbb{N}$. Firstly, we will associate a density $\mathcal{D}_n^{(n)}$ to each interval $I_j^n \in \mathfrak{D}_n$ in order for it to be a $\mathcal{Z}$-1 sequence. Specifically, $\mathcal{D}_n^{(n)}$ will be constructed as the maximal $\mathcal{Z}$-1 sequence such that if $I_j^n \cap K = \varnothing$, then $\mathcal{D}_n^{(n)}(I_j^n) = 0$. Secondly, we will associate, to each interval $I_j^{n-1} \in \mathfrak{D}_{n-1}$, the maximal density $\mathcal{D}_{n-1}^{(n)}$ such that the resulting sequence is $\mathcal{Z}$-2 and that $\mathcal{D}_{n-1}^{(n)}(I_j^{n-1}) \leq \mathcal{D}_n^{(n)}(I_{2j}^n) + \mathcal{D}_n^{(n)}(I_{2j+1}^n)$. Iterating this process we will obtain a density $\mathcal{D}_0^{(n)}([0,1])$. The limit of $2^{-n}\mathcal{D}_0^{(n)}([0,1])$ as $n \to \infty$ bounds the maximum mass a Zygmund measure defined on the intervals of $\mathfrak{D}_n$ that intersect K and with controlled $\|\mu\|_*$ can have.

Let us formally define the densities $\mathcal{D}_k^{(n)}$ associated to each dyadic interval in $\mathfrak{D}_k$ for $k \leq n$. We start by defining $\mathcal{D}_n^{(n)}$. Given an interval $I_j^n \in \mathfrak{D}_n$ for $j = 0, 1, \dots, 2^n - 1$, we define

$$\mathcal{D}_n^-(I_j^n) = \begin{cases} 0 & \text{if } I_j^n \cap K = \emptyset, \\ \mathcal{D}_n^-(I_{j-1}^n) + 1 & \text{if } I_j^n \cap K \neq \emptyset, \end{cases}$$

and analogously,

$$\mathcal{D}_n^+(I_j^n) = \begin{cases} 0 & \text{if } I_j^n \cap K = \emptyset, \\ \mathcal{D}_n^+(I_{j+1}^n) + 1 & \text{if } I_j^n \cap K \neq \emptyset, \end{cases}$$

with the convention $I_{-1}^n = \left[\frac{-1}{2^n}, 0\right)$, $I_{2^n}^n = \left[1, 1 + \frac{1}{2^n}\right)$ and $\mathcal{D}_n^-(I_{-1}^n) = \mathcal{D}_n^+(I_{2^n}^n) = 0$. Finally, we denote

$$\mathcal{D}_n^{(n)}(I_j^n) = \min\{\mathcal{D}_n^-(I_j^n), \mathcal{D}_n^+(I_j^n)\}.$$

Now we define $\mathcal{D}_{n-k}(I_j^{n-k})$ for each dyadic interval $I_j^{n-k} \in \mathfrak{D}_{n-k}$ for $1 \leq k \leq n$. To do so, let us consider the two intervals $J, J' \in \mathfrak{D}_{n-k+1}$ such that $J, J' \subseteq I_j^{n-k}$. We denote

$$S_{n-k}(I_j^{n-k}) = \mathcal{D}_{n-k+1}^{(n)}(J) + \mathcal{D}_{n-k+1}^{(n)}(J'),$$

and we proceed as before, setting

$$\mathcal{D}_{n-k}^{-}(I_j^{n-k}) = \min\{\mathcal{D}_{n-k}^{-}(I_{j-1}^{n-k}) + 2^k,\ S_{n-k}(I_j^{n-k})\},$$
$$\mathcal{D}_{n-k}^{+}(I_j^{n-k}) = \min\{\mathcal{D}_{n-k}^{+}(I_{j+1}^{n-k}) + 2^k,\ S_{n-k}(I_j^{n-k})\},$$

with the convention $\mathcal{D}_{n-k}^{-}(I_{-1}^{n-k}) = \mathcal{D}_{n-k}^{+}(I_{2^{n-k}}^{n-k}) = 0$. Finally, we define

$$\mathcal{D}_{n-k}^{(n)}(I_j^{n-k}) = \min\{\mathcal{D}_{n-k}^{-}(I_j^{n-k}),\ \mathcal{D}_{n-k}^{+}(I_j^{n-k})\}.$$

Hence, we built the densities $\mathcal{D}_n^{(n)}, \mathcal{D}_{n-1}^{(n)}, \dots, \mathcal{D}_0^{(n)}$. Finally, we define the *Zygmund Capacity* as

$$C_n(K) = 2^{-n}\,\mathcal{D}_0^{(n)}([0,1]).$$

Let K' be the compact set formed by the union of the dyadic intervals in $\mathfrak{D}_n$ which intersect K. By construction, $C_n(K)$ is an upper bound to the mass of any Zygmund measure with support in K'.

Proposition 3.3. *For each compact set $K \subset [0,1]$, $\exists \lim_{n\to\infty} C_n(K)$.*

Proof. By construction, $C_n(K) \geq 0\ \forall n$. In order to prove that the limit exists, it suffices to see that the succession $(C_n(K))_n$ is decreasing. Let $I_j^n \in \mathfrak{D}_n$ be an interval such that $I_j^n \cap K = \emptyset$ and let I_{2j}^{n+1}, I_{2j+1}^{n+1} be the two intervals of $\mathfrak{D}_{n+1}$ contained in I_j^n. Clearly, I_{2j}^{n+1} and I_{2j+1}^{n+1} are disjoint from K. Hence,

$$\mathcal{D}_n^{(n)}(I_j^n) = 0 \Longrightarrow \mathcal{D}_n^{(n+1)}(I_j^n) = 0.$$

Alternatively, if $I_j^n \cap K \neq \emptyset$, then $\mathcal{D}_n^{(n)}(I_j^n) = a > 0$. Therefore, we conclude that $\mathcal{D}_n^{(n+1)}(I_{j-a}^n) = 0$ or that $\mathcal{D}_n^{(n+1)}(I_{j+a}^n) = 0$. Consequently, $\mathcal{D}_n^{(n+1)}(I_j^n) \leq 2a$. This implies that $\mathcal{D}_n^{(n+1)}(I_j^n) \leq 2\mathcal{D}_n^{(n)}(I_j^n)$ for all j, so $\mathcal{D}_0^{(n+1)}([0,1]) \leq 2\mathcal{D}_0^{(n)}([0,1])$. As a consequence, $C_{n+1}(K) \leq C_n(K)$. □

Theorem 3.4. *Let $K \subseteq [0,1]$ a compact set. If $\lim_{n\to\infty} C_n(K) = 0$, K cannot be the support of a nontrivial Zygmund measure.*

Proof. Let μ be a Zygmund measure with support in K. We will prove that μ must be the trivial measure. Let us assume, without loss of generality, that $\|\mu\|_* \leq 1$. Given $n \in \mathbb{N}$, let $0 \leq k \leq n$ be an integer. We will prove by induction on k that $\mu(I_j^{n-k}) \leq 2^{-n}\,\mathcal{D}_{n-k}^{(n)}(I_j^{n-k})$ for all $I_j^{n-k} \in \mathfrak{D}_{n-k}$.

It is clear that if $I_j^{n-k} \cap K = \emptyset$, the inequality holds, so let us assume that $I_j^{n-k} \cap K \neq \emptyset$.

Firstly, we will show that the inequality holds for $k = 0$. Let $I_j^n \in \mathfrak{D}_n$ be a dyadic interval and let $I_{j-\ell}^n \in \mathfrak{D}_n$ be the closest interval to I_j^n such that $I_{j-\ell}^n \cap K = \emptyset$. Note that $-2^n + j \leq \ell \leq j+1$ and we can assume, without loss of generality that $\ell > 0$. Therefore,

$$\frac{\mu(I_j^n)}{|I_j^n|} \leq 1 + \frac{\mu(I_{j-1}^n)}{|I_j^n|} \leq 1 + 1 + \frac{\mu(I_{j-2}^n)}{|I_j^n|} \leq \cdots \leq \ell + \frac{\mu(I_{j-\ell}^n)}{|I_j^n|} = \ell = \mathcal{D}_n^{(n)}(I_j^n)$$

and $\mu(I_j^n) \leq 2^{-n}\mathcal{D}_n^{(n)}(I_j^n)$. Let us assume the inequality holds for $n-k+1$ where $1 \leq k \leq n$ is fixed, and we will prove it holds for $n-k$. On one hand we have

$$\mu(I_j^{n-k}) = \mu(I_{2j}^{n-k+1}) + \mu(I_{2j+1}^{n-k+1}) \leq 2^{-n}(\mathcal{D}_{n-k+1}^{(n)}(I_{2j}^{n-k+1}) + \mathcal{D}_{n-k+1}^{(n)}(I_{2j+1}^{n-k+1})),$$

since μ is a measure and $I_j^{n-k} = I_{2j}^{n-k+1} \cup I_{2j+1}^{n-k+1}$. On the other hand,

$$\frac{\mu(I_j^{n-k})}{|I_j^{n-k}|} \leq 1 + \frac{\mu(I_{j-1}^{n-k})}{|I_j^{n-k}|} \Longrightarrow \mu(I_j^{n-k}) \leq 2^{-n+k} + \mu(I_{j-1}^{n-k}).$$

Analogously, $\mu(I_j^{n-k}) \leq 2^{-n+k} + \mu(I_{j+1}^{n-k})$. As both of these inequalities hold for all j, clearly

$$\mu(I_j^{n-k}) \leq 2^{-n}\min\{\mathcal{D}_{n-k+1}^{(n)}(I_{2j}^{n-k+1}) + \mathcal{D}_{n-k+1}^{(n)}(I_{2j+1}^{n-k+1}), \mathcal{D}_{n-k}^{-}(I_{j-1}^{n-k}) + 2^k, \mathcal{D}_{n-k}^{+}(I_{j+1}^{n-k}) + 2^k\}.$$

Therefore, $\mu(I_j^{n-k}) \leq 2^{-n}\mathcal{D}_{n-k}^{(n)}(I_j^{n-k})$ which implies that $\mu([0,1]) \leq 2^{-n}\mathcal{D}_0^{(n)}([0,1]) = C_n(K) \xrightarrow{n\to\infty} 0$. Consequently, since $K \subset [0,1]$ we have $\mu(K) = 0$ and, as a result, $\mu \equiv 0$. □

Conjecture 3.5. *A compact set K is the support of a nontrivial Zygmund measure if and only if*

$$\lim_{n\to\infty} C_n(K) > 0.$$

By Theorem 3.4, it suffices to prove that if the limit of $C_n(K)$ as n tends to infinity is positive, then there exists a nontrivial Zygmund measure with support in K. In order to do so, we will define a succession of Zygmund measures μ_n with mass equal to $C_n(K)$. As $\lim\limits_{n\to\infty} \|\mu_n\| = \lim_{n\to\infty} C_n(K) > 0$, the limit $\mu = \lim\limits_{n\to\infty} \mu_n$ will be well-defined. Choosing appropriately the support of μ_n, μ will have support in K. Furthermore, if μ_n are uniformly bounded, then μ will be a Zygmund measure.

Given an integer n we denote $\mathcal{D}_0^{(n)} := \mathcal{D}_0^{(n)}([0,1])$. By construction, $\mathcal{D}_0^{(n)} \leq \sum\limits_{j=0}^{2^n-1} \mathcal{D}_n^{(n)}(I_j^n)$ where $I_j^n \in \mathfrak{D}_n$. If both expressions are equal, then considering

$$\mathrm{d}\mu_n = \sum_{j=0}^{2^n-1} \mathcal{D}_n^{(n)}(I_j^n)\chi_{I_j^n}\,\mathrm{d}x,$$

we have that μ_n is a Zygmund measure. Let us assume, that $\mathcal{D}_0^{(n)} < \sum_{j=0}^{2^n-1} \mathcal{D}_n^{(n)}(I_j^n)$. We want to find a Zygmund measure μ_n with mass $2^{-n}\mathcal{D}_0^{(n)}$, constant over dyadic intervals I_j^n and such that $\mu_n(I_j^n) = 0$ if I_j^n is disjoint from K.

Therefore we aim to determine some numbers $d_0, \dots, d_{2^n-1} \geq 0$ such that $\sum\limits_{j=0}^{2^n-1} d_j = \mathcal{D}_0^{(n)}$, that

$$\mathrm{d}\mu_n = \sum_{j=0}^{2^n-1} d_j\chi_{I_j^n}\,\mathrm{d}x \tag{3}$$

is a Zygmund measure and that $d_j = 0$ if $I_j^n \cap K = \emptyset$. Finding a technique to determine this numbers would end the proof, since the limit of μ_n would be a positive Zygmund measure with support in K.

Therefore, we will try to *distribute* $\mathcal{D}_0^{(n)}$ between $d_0, \dots, d_{2^n-1} \geq 0$ in such a way that the aforementioned conditions are met. Clearly, in order for μ_n as defined in (3) to be a Zygmund measure with $\|\mu\|_* \leq 1$, the sequence $d_0 d_1 \dots d_{2^n-1}$ must be $\mathcal{Z}$-1. By construction, the sequence $\mathcal{D}_n^{(n)}(I_0^n) \dots \mathcal{D}_n^{(n)}(I_{2^n-1}^n)$ is the maximal $\mathcal{Z}$-1 sequence such that the j-th number is zero if $I_j^n \cap K = \emptyset$. As a result, we have $d_j \leq \mathcal{D}_n^{(n)}(I_j^n)$ for all j. Let us denote $d_j^1 = d_{2j} + d_{2j+1}$. Then, in order for (3) to be a Zygmund measure, the sequence $d_0^1 \dots d_{2^{n-1}-1}^1$ must be $\mathcal{Z}$-2 and less than $\mathcal{D}_{n-1}^{(n)}(I_0^{n-1}) \dots \mathcal{D}_{n-1}^{(n)}(I_{2^{n-1}-1}^{n-1})$. Denoting, inductively, $d_j^k = d_{2j}^{k-1} + d_{2j+1}^{k-1}$ for $1 \leq k \leq n$ we conclude that the sequences $d_0^k \dots d_{2^{n-k}-1}^k$ must be $\mathcal{Z}$-2^k and less than $\mathcal{D}_{n-k}^{(n)}(I_0^{n-k}) \dots \mathcal{D}_{n-k}^{(n)}(I_{2^{n-k}-1}^{n-k})$. Note that $d_0^n = \mathcal{D}_0^{(n)}$.

We shall say that $d_0, \dots, d_{2^n-1}$ constitute a *distribution* of $\mathcal{D}_0^{(n)}$ if they meet the following conditions:

- $d_j \geq 0$ for all j,
- $\sum_{j=0}^{2^n-1} d_j = \mathcal{D}_0^{(n)}$,
- $d_0^k \dots d_{2^{n-k}-1}^k$ must be a $\mathcal{Z}$-2^k sequence for all $k = 0, \dots, n$,
- $d_j^k \leq \mathcal{D}_{n-k}^{(n)}(I_j^{n-k})$ for all $k = 0, \dots, n$ and for all $j = 0, \dots, 2^{n-k} - 1$,

where $d_j^0 = d_j$ for all j.

If $d_0, \dots, d_{2^n-1}$ is a *distribution* of $\mathcal{D}_0^{(n)}([0,1])$, then the measure defined in (3) meets the hypothesis of Lemma 3.1. This implies that μ_n is a Zygmund measure with mass $\|\mu_n\| = 2^{-n}\mathcal{D}_0^{(n)}([0,1]) = C_n(K)$.

First of all, we are going to prove that given d_j^k for $j = 0, \dots, 2^{n-k} - 1$ that meet certain conditions, we can determine d_j^{k-1} for $j = 0, \dots, 2^{n-k+1} - 1$. After that, we will state a *distribution* method that appears to guarantee that the conditions are met, although we have not been able to prove this result. To that end, let us introduce the concept of sequence of corrected sums.

Definition 3.6. Let $n \in \mathbb{N}$ and $1 \leq k \leq n$. Given an integer $\mathcal{Z}$-2^{k-1} sequence $x_1\, x_2\, \dots\, x_{2^{n-k+1}}$, we determine its *sequence of corrected sums* $t_1\, t_2\, \dots\, t_{2^{n-k}}$ as:

- Firstly, let us consider the integer sequence $s_1\, s_2\, \dots\, s_{2^{n-k}}$ defined as $s_j = x_{2j-1} + x_{2j}\ \forall j$.
- Secondly, let $e_1\, e_2\, \dots\, e_{2^{n-k}}$ and $d_1\, d_2\, \dots\, d_{2^{n-k}}$ be two integer sequences constructed as follows:

$$\begin{cases} e_1 = \min\{s_1,\, 2^k\}, \\ d_{2^{n-k}} = \min\{s_{2^{n-k}},\, 2^k\}, \\ e_j = \min\{s_j,\, e_{j-1} + 2^k\} & j = 2, \dots, 2^{n-k}, \\ d_j = \min\{s_j,\, d_{j+1} + 2^k\} & j = 1, \dots, 2^{n-k} - 1. \end{cases}$$

- Finally, we define the integer sequence $t_1\, t_2\, \dots\, t_{2^{n-k}}$ as $t_j = \min\{e_j, d_j\}\ \forall j$.

Note that, given a sequence $\mathcal{Z}$-2^{k-1}, its sequence of corrected sums will be $\mathcal{Z}$-2^k.

Lemma 3.7. *Given $n \in \mathbb{N}$ and $1 \leq k \leq n$, let $x_1 x_2 \dots x_{2^{n-k+1}}$ be an integer $\mathcal{Z}$-2^{k-1} sequence and let us denote by $t_1 t_2 \dots t_{2^{n-k}}$ its sequence of corrected sums. Let $r_1 r_2 \dots r_{2^{n-k}}$ be an integer $\mathcal{Z}$-2^k sequence with $r_j \leq t_j$ $\forall j$. Then there exists $y_1 y_2 \dots y_{2^{n-k+1}}$ an integer $\mathcal{Z}$-2^{k-1} sequence satisfying $y_j \leq x_j$ for $j = 1, \dots, 2^{n-k+1}$ and $y_{2j-1} + y_{2j} = r_j$ for $j = 1, \dots, 2^{n-k}$, if the following conditions are met*

$$\begin{cases} -x_{2j-3} - 2^{k-1} \leq r_j - r_{j-1} \leq x_{2j} + 2^{k-1}, \\ r_j - r_{j-1} \leq 2^k + x_{2j} - y_{2j-4} \end{cases}$$

for all $j = 2, \dots, 2^{n-k}$.

Notation. Let us introduce the following notation: $a_j = x_{2j} - x_{2j-1}$, $\delta_j = x_{2j+1} - x_{2j}$, $b_j = s_j - r_j$ and $\tilde{b}_j = x_j - y_j$.

Proof. We will construct the sequence $y_1 y_2 \dots y_{2^{n-k+1}}$ as $y_j = x_j - \tilde{b}_j$ where $\tilde{b}_{2j-1} + \tilde{b}_{2j} = b_j$. We will take $\tilde{b}_j \in \mathbb{N}$, as a result, $y_j \leq x_j$, $y_j \in \mathbb{N}$ and $y_{2j-1} + y_{2j} = s_j - b_j = r_j$. Consequently, we only need to prove that there exists $\tilde{b}_j$ with the previous definition such that $y_1 y_2 \dots y_{2^{n-k+1}}$ satisfies $y_j \geq 0$ $\forall j$ and $|y_j - y_{j-1}| \leq 2^{k-1}$ for $j = 2, \dots, 2^{n-k+1}$.

Note that $\tilde{b}_{2j-1}$ needs to satisfy the following conditions:

(i) $\tilde{b}_{2j-1} \in I_0 = [b_j - x_{2j},\ x_{2j-1}]$ since $y_j \geq 0$ implies $\tilde{b}_{2j-1} \leq x_{2j-1}$ and $\tilde{b}_{2j} = b_j - \tilde{b}_{2j-1} \leq x_{2j}$.

(ii) $\tilde{b}_{2j-1} \in I_1 = [0, b_j]$ since $y_j \leq x_j$ implies $\tilde{b}_{2j-1} \geq 0$ and $\tilde{b}_{2j-1} = b_j - \tilde{b}_{2j}$.

(iii) $\tilde{b}_{2j-1} \in I_2 = \left[\left\lceil \frac{b_j - a_j}{2} \right\rceil - 2^{k-2}, \left\lfloor \frac{b_j - a_j}{2} \right\rfloor + 2^{k-2}\right]$, where $\lfloor \ell \rfloor$ and $\lceil \ell \rceil$ denote the floor and ceiling of ℓ respectively. This condition derives from the inequality $|y_{2j} - y_{2j-1}| \leq 2^{k-1}$.

(iv) $\tilde{b}_{2j-1} \in I_3 = [\delta_{j-1} + \tilde{b}_{2j-2} - 2^{k-1},\ \delta_{j-1} + \tilde{b}_{2j-2} + 2^{k-1}]$. This condition originates derives from the fact that

$$|y_{2j-1} - y_{2j-2}| = |x_{2j-1} - \tilde{b}_{2j-1} - x_{2j-2} + \tilde{b}_{2j-2}| = |\delta_{j-1} - \tilde{b}_{2j-1} + \tilde{b}_{2j-2}| \leq 2^{k-1}.$$

The remainder of the proof consists on checking that $I_0 \cap I_1 \cap I_2 \cap I_3 \neq \varnothing$ when the conditions of the lemma are met. □

Notation. Let us denote by I_j the intersection of the aforementioned four intervals of the j-th step. By construction, the endpoints of I_j are integer, and we will denote them by $I_j = [\tilde{b}_{2j-1,m}, \tilde{b}_{2j-1,M}]$, with

$$\tilde{b}_{2j-1,m} = \max\left\{b_j - x_{2j}, 0, \left\lceil \frac{b_j - a_j}{2} \right\rceil - 2^{k-2}, \delta_{j-1} + \tilde{b}_{2j-2} - 2^{k-1}\right\},$$

$$\tilde{b}_{2j-1,M} = \min\left\{x_{2j-1}, b_j, \left\lfloor \frac{b_j - a_j}{2} \right\rfloor + 2^{k-2}, \delta_{j-1} + \tilde{b}_{2j-2} + 2^{k-1}\right\}.$$

Note that Lemma 3.7 implies that $I_j \neq \emptyset$. As a result, $\tilde{b}_{2j-1,m} \leq \tilde{b}_{2j-1,M}$ if the conditions are met. Let $\tilde{b}_{2j-1} \in I_j$ be an integer which minimises the distance[2]

$$d(\tilde{b}_{2j-1}) = \max\{|b_j - a_j - 2\tilde{b}_{2j-1}|, |y_{2j-2} - x_{2j-1} + \tilde{b}_{2j-1}|\}$$

subject to the restriction

$$\delta_j + b_j - \tilde{b}_{2j-1} + 2^{k-1} \geq \min\left\{x_{2j+1}, b_{j+1}, \left\lfloor \frac{b_{j+1} - a_{j+1}}{2} \right\rfloor + 2^{k-2}, \delta_j + b_j - \tilde{b}_{2j-1,w} + 2^{k-1}\right\} = \tilde{b}_{2j+1,M}$$

if $r_{j+1} \leq r_{j+2}$; or subject to the restriction

$$\delta_j + b_j - \tilde{b}_{2j-1} - 2^{k-1} \leq \max\left\{b_{j+1} - x_{2j+2}, 0, \left\lceil \frac{b_{j+1} - a_{j+1}}{2} \right\rceil - 2^{k-2}, \delta_j + b_j - \tilde{b}_{2j-1,w} - 2^{k-1}\right\} = \tilde{b}_{2j+1,m}$$

if $r_{j+1} > r_{j+2}$, where

$$\tilde{b}_{2j-1,w} = \begin{cases} \tilde{b}_{2j-1,M} & \text{if } r_j \leq r_{j+1}, \\ \tilde{b}_{2j-1,m} & \text{if } r_j > r_{j+1}. \end{cases}$$

If $j = 2^{n-k}$, the restriction to be satisfied is $x_{2j} - b_j + \tilde{b}_{2j-1} \leq 2^{k-1}$.

It can be seen easily that the aforementioned restrictions and Lemma 3.7 guarantee that the intersection of the intervals will not be empty.

Given two sequences r_j and x_j that satisfy the hypothesis of Lemma 3.7, we apply the aforemention method to obtain an integer sequence, denoted by $\tilde{b}_j^{(e)}$. Then we build a sequence symbolised by $y^{(e)}$, as $y_j^{(e)} = x_j - \tilde{b}_j^{(e)}$.

Let $r_j' = r_{2^{n-k}} \dots r_2 r_1$ and $x_j' = x_{2^{n-k+1}} \dots x_2 x_1$, be the inverted sequences r_j and x_j respectively. Clearly both sequences meet the conditions of Lemma 3.7. Applying the previously mentioned method, we obtain another integer sequence, $\tilde{b}_j^{(d)}$. Then, we build the sequence $y^{(d)}$, as $y_j^{(d)} = x_j - \tilde{b}^{(d)}_{2^{n-k+1}-j}$.

Note that $y^{(e)}$ and $y^{(d)}$ might not be $\mathcal{Z}$-2^{k-1} sequences, however, the following linear combination will be. We define y_j as

$$\begin{cases} \begin{cases} y_{2j-1} = \dfrac{y_{2j-1}^{(e)} + y_{2j-1}^{(d)}}{2} \\ y_{2j} = \dfrac{y_{2j}^{(e)} + y_{2j}^{(d)}}{2} \end{cases} & \text{if } (y_{2j-1}^{(e)} + y_{2j-1}^{(d)}) \equiv 0 \pmod 2, \\ \begin{cases} y_{2j-1} = \dfrac{y_{2j-1}^{(e)} + y_{2j-1}^{(d)} + 1}{2} \\ y_{2j} = \dfrac{y_{2j}^{(e)} + y_{2j}^{(d)} - 1}{2} \end{cases} & \text{if } (y_{2j-1}^{(e)} + y_{2j-1}^{(d)}) \equiv 1 \pmod 2 \quad \text{and } \Delta_1 < \Delta_2, \\ \begin{cases} y_{2j-1} = \dfrac{y_{2j-1}^{(e)} + y_{2j-1}^{(d)} - 1}{2} \\ y_{2j} = \dfrac{y_{2j}^{(e)} + y_{2j}^{(d)} + 1}{2} \end{cases} & \text{if } (y_{2j-1}^{(e)} + y_{2j-1}^{(d)}) \equiv 1 \pmod 2 \quad \text{and } \Delta_1 \geq \Delta_2, \end{cases}$$

[2] If $j = 1$ we take $y_{2j-2} = 0$.

where

$$\Delta_1 = \max\left\{ \left| \frac{y^{(e)}_{2j-1} + y^{(d)}_{2j-1} + 1}{2} - y_{2j-2} \right|, \left| \frac{y^{(e)}_{2j} + y^{(d)}_{2j} - 1}{2} - \frac{y^{(e)}_{2j-1} + y^{(d)}_{2j-1} + 1}{2} \right|, \left| \frac{y^{(e)}_{2j+1} + y^{(d)}_{2j+1}}{2} - \frac{y^{(e)}_{2j} + y^{(d)}_{2j} - 1}{2} \right| \right\},$$

$$\Delta_2 = \max\left\{ \left| \frac{y^{(e)}_{2j-1} + y^{(d)}_{2j-1} - 1}{2} - y_{2j-2} \right|, \left| \frac{y^{(e)}_{2j} + y^{(d)}_{2j} + 1}{2} - \frac{y^{(e)}_{2j-1} + y^{(d)}_{2j-1} - 1}{2} \right|, \left| \frac{y^{(e)}_{2j+1} + y^{(d)}_{2j+1}}{2} - \frac{y^{(e)}_{2j} + y^{(d)}_{2j} + 1}{2} \right| \right\}.$$

Although this method yielded promising numerical results, we have not been able to prove that this method of *distribution* guarantees that the conditions of Lemma 3.7 are always met.

References

[1] J.J. Carmona, J.J. Donaire, "On removable singularities for the analytic Zygmund class", *Michigan Math. J.* **43**(1) (1996), 51–65.

[2] J. Garnett, *Analytic Capacity and Measure*, Lecture Notes in Mathematics **297**, Springer-Verlag, Berlin-New York, 1972.

[3] J.-P. Kahane, "Trois notes sur les ensembles parfaits linéaires", *Enseign. Math. (2)* **15** (1969), 185–192.

[4] R.P. Kaufman, "Smooth functions and porous sets", *Proc. Roy. Irish Acad. Sect. A* **93**(2) (1993), 189–191.

[5] N.G. Makarov, "Probability methods in the theory of conformal mappings" (Russian), *Algebra i Analiz* **1**(1) (1989), 3–59; translation in: *Leningrad Math. J.* **1**(1) (1990), 1–56.

[6] N.G. Makarov, "A class of exceptional sets in the theory of conformal mappings" (Russian), *Mat. Sb.* **180**(9) (1989), 1171–1182, 1296; translation in: *Math. USSR-Sb.* **68**(1) (1991), 19–30.

[7] P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces*, Fractals and rectifiability, Cambridge Studies in Advanced Mathematics **44**, Cambridge University Press, Cambridge, 1995.

[8] A. Zygmund, "Smooth functions", *Duke Math. J.* **12** (1945), 47–76.