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Homotopical realizations of infinity groupoids

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Resum *(CAT)*
La hipòtesi d'homotopia de Grothendieck afirma que l'estudi dels tipus d'homotopia dels espais topològics és equivalent a l'estudi dels ∞-grupoides. En la pràctica, un cop triat un model per a les categories d'ordre superior, l'equivalència és realitzada per l'assignació del ∞-grupoide fonamental a un espai topològic. Proposem un model accessible per al ∞-grupoide fonamental, usant categories topològiques per a modelitzar els ∞-grupoides.

Abstract *(ENG)*
Grothendieck's homotopy hypothesis asserts that the study of homotopy types of topological spaces is equivalent to the study of ∞-groupoids. In practice, after one has chosen a model for higher categories, the equivalence is realized by the assignment of the fundamental ∞-groupoid to a topological space. We propose an accessible model for the fundamental ∞-groupoid, using topological categories to model ∞-groupoids.

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1. Introduction

This article results from the study of an equivalence between topological spaces and ∞-groupoids in the framework of homotopy theory: the homotopy hypothesis. This equivalence was described in the bachelor thesis of the author [7], and as done there, we now use it to give a more manageable model for the *fundamental ∞-groupoid* of a topological space in that context. We will after study when this assignment also gives such an equivalence.

What is algebraic topology?

The fundamental concept in this area of mathematics is that of homotopy, which is a notion of continuous "deformation". Concretely, two maps are homotopic if we can "deform" one into the other in a continuous manner. From here stems the notion of (weak) homotopy equivalence between topological spaces; for example, the plane without the origin is homotopy equivalent to the circle. The goal of algebraic topology is to classify the homotopy types of topological spaces, where we say that two spaces are of the same homotopy type if they are homotopy equivalent. To do this, invariants under homotopy equivalence are studied, such as the fundamental group. The fundamental group is a very useful invariant to classify homotopy types, but is far from allowing a complete classification.

Models for the homotopy types

A more natural way of packaging the information of the fundamental group is to consider the fundamental groupoid, since we do not need to make any choice of basepoint. The *fundamental groupoid* is a category whose objects are the points of the topological space and whose morphisms are homotopy classes of paths.

It is not surprising that this object does not permit us to model the homotopy types completely, because we discard a lot of information when considering the paths up to homotopy. Therefore we will consider paths as morphisms and keep the information about the homotopies separately. We will consider homotopies as 2-morphisms, that is morphisms between morphisms. In the same way, we will consider homotopies between homotopies as 3-morphisms, etc.

The object obtained when taking into account all the higher homotopies is the so-called *fundamental ∞-groupoid*. This is a genuine idea of ∞-groupoid, with which Grothendieck suggested that there should exist a good model for the notion of ∞-groupoid for which the study of the homotopy types of topological spaces is equivalent to that of ∞-groupoids. This is what is generally known as the homotopy hypothesis, and it is seen as a proof for the suitability of a model of ∞-groupoids. The ideas of Grothendieck are found in his manuscript "À la poursuite des champs" [1], where the mathematician started a search for such a model.

Results

Once equipped with some ideas in this introduction we can better make sense of the results involved in this article. We will model ∞-groupoids by certain topological categories. The way in which we proved in [7] the mentioned equivalence is by giving an equivalence of *homotopy categories*, a setting in which homotopy equivalences, or more generally weak equivalences, become isomorphisms. The result in [7] was:

Theorem A. *The homotopy category of topological spaces is equivalent to the homotopy category of ∞-groupoids, in the sense of Definition 3.3. This equivalence is realized by a composition of functors $|\mathfrak{C}(\mathrm{Sing}(-))|$ and its inverse $|\mathfrak{N}(\mathrm{Sing}(-))|$, introduced in Section 5.*

This is based on the theory of model categories and the classical homotopy-theoretic equivalence of topological spaces with simplicial sets in [8]; Theorem 2.2.5.1 in [5] – also due to Joyal; and [4]. We will use this to proof our main result, which says that the model that we propose for the fundamental ∞-groupoid has the homotopy type it should have, according to Theorem A:

Theorem B. *The ∞-groupoid $\Pi(X)$ constructed in Construction 4.1 is weakly equivalent to $|\mathfrak{C}(\mathrm{Sing}(X))|$ for every topological space X.*

Finally, we describe a situation in which the functor Π from the category of topological spaces to the category of ∞-groupoids gives an equivalence of homotopy categories. We obtained the following, based on the classical equivalence between connected pointed topological spaces and group-like $\mathbb{E}_1$-spaces – see Section 6 for the notation:

Theorem C. *The fundamental ∞-groupoid functor Π and the classifying space functor B induce an equivalence of categories $\mathrm{Ho}(\mathbf{Top}_*^{\geq 0}) \simeq \mathrm{Ho}(\infty\text{-}\mathbf{Grpd}_*^{\geq 0})$.*

2. Categories with weak equivalences

We start with a first hint about the existence of homotopy theory in more general contexts than topological spaces. A *category with weak equivalences* is a category $\mathcal{C}$ equipped with a class of morphisms W of $\mathcal{C}$ which contains all isomorphisms of $\mathcal{C}$, and satisfy the two-out-of-three property: for f and g any two composable morphisms in $\mathcal{C}$, if two of $\{f, g, g \circ f\}$ are in W, then so is the third.

We will define the homotopy category of $\mathcal{C}$ by a universal property which can more generally described as follows. Let $\mathcal{C}$ be an arbitrary category and let S be a subclass of the class of maps of $\mathcal{C}$. By the *localization* of $\mathcal{C}$ with respect to S we mean a category $S^{-1}\mathcal{C}$ together with a functor $\gamma\colon \mathcal{C} \longrightarrow S^{-1}\mathcal{C}$ having the following universal property. For every $s \in S$, $\gamma(s)$ is an isomorphism; given any functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ with $F(s)$ an isomorphism for all $s \in S$, there is a unique functor $\theta\colon S^{-1}\mathcal{C} \longrightarrow \mathcal{D}$ such that $\theta \circ \gamma = F$.

Definition 2.1. Let $(\mathcal{C}, W)$ be a category with weak equivalences. Then the *homotopy category* of $\mathcal{C}$ is the localization of $\mathcal{C}$ with respect to the class W of weak equivalences and is denoted by $\gamma\colon \mathcal{C} \longrightarrow \mathrm{Ho}(\mathcal{C})$.

If it exists, the homotopy category is the category obtained from $\mathcal{C}$ by formally inverting all the weak equivalences. Proving existence can entail set-theoretical difficulties. To guarantee the existence one usually relies on extra structure, such as a *model category* structure. This is our case.

For example, the category of compactly generated topological spaces, denoted by **Top**, has a model structure [3, §2.4]. The weak equivalences are the *weak homotopy equivalences*: those maps which induce an isomorphism on each homotopy group and a bijection between path-components.

3. ∞-groupoids as topological categories

We now enter the context in which we will define ∞-groupoids: enriched categories – see [5, A.1.3, A.1.4] for details. The cases that interest us are categories enriched over topological spaces and over simplicial sets. These are "categories" in which there is a notion of morphisms of every order.

Definition 3.1. A *topological category* is a category enriched over **Top**, the category of compactly generated topological spaces.

In a topological category $\mathcal{C}$, for each pair of objects x and y we have a topological space $\mathrm{Map}_{\mathcal{C}}(x, y)$; we think of its points as 1-morphisms, and of a path in it as a 2-morphism. In general, for $n > 1$, an n-morphism in $\mathcal{C}$ is a homotopy between two $(n-1)$-morphisms inside the space $\mathrm{Map}_{\mathcal{C}}(x, y)$.

We denote the category of topological categories and topologically enriched functors by $\mathbf{Cat_{top}}$. We have a functor $\pi_0 \colon \mathbf{Cat_{top}} \longrightarrow \mathbf{Cat}$ to ordinary categories by taking path-components of the morphism spaces. Continuing with the ideas in the previous paragraph, π_0 has the effect of taking homotopy classes of 1-morphisms in $\mathcal{C}$, and discarding all higher homotopies. For a topological category $\mathcal{C}$, we call $\pi_0\mathcal{C}$ its *homotopy category*.

Definition 3.2. Let $F \colon \mathcal{C} \longrightarrow \mathcal{D}$ be a functor between topological categories. We say that F is a *weak equivalence* if the following conditions hold:

- For every pair of objects $x, y \in \mathcal{C}$, the induced map
$$\mathrm{Map}_{\mathcal{C}}(x, y) \longrightarrow \mathrm{Map}_{\mathcal{D}}(Fx, Fy)$$
is a weak homotopy equivalence of topological spaces.
- Every object of $\mathcal{D}$ is isomorphic in $\pi_0\mathcal{D}$ to Fx, for some $x \in \mathcal{C}$.

The analogy with topological spaces is clear: the given definition is a weak version of the notion of equivalence of categories, in the same way in which weak homotopy equivalences are with respect to homotopy equivalences.

We now turn our attention to the most central definition in our work. Recall that a *groupoid* is a category in which every morphism in invertible. The following characterizes those topological categories in which every morphism is invertible up to higher order morphisms or, in other words, in which every morphism has a *homotopy inverse*.

Definition 3.3. Let $\mathcal{C}$ be a topological category. We say that $\mathcal{C}$ is an *∞-groupoid* if $\pi_0\mathcal{C}$ is a groupoid.

The condition on $\pi_0\mathcal{C}$ guarantees that for every 1-morphism f of $\mathcal{C}$ there exists a 1-morphism g, in the opposite direction, such that $g \circ f$ and $f \circ g$ are in the same path-component as the identity in the corresponding endomorphism spaces. In other words, every 1-morphism has a *homotopy inverse*. Every morphism of order greater than 1 in a topological category is a homotopy in some topological space, and therefore it is already invertible up to homotopy.

The category of ∞-groupoids and topologically enriched functors will be denoted by ∞-**Grpd**; it is a full subcategory of $\mathbf{Cat_{top}}$. We may consider its homotopy category, with the weak equivalences being the ones defined in Definition 3.2, because of [4].

Recall that a *topological monoid* is a monoid with respect to which the binary operation is a continuous map. A useful observation is that the structure of a topological category with one object is just the structure of a topological monoid on the space of endomorphisms of the single object; just as for ordinary categories and monoids.

A characteristic property of ∞-groupoids, which will feature in the proofs of Theorems B and C, is that every connected ∞-groupoid is weakly equivalent to a *group-like* topological monoid, i.e., a topological monoid $\mathcal{M}$ such that $\pi_0\mathcal{M}$ is a group. The precise statement is the following:

Lemma 3.4. *Let $\mathcal{G}$ be a connected ∞-groupoid. Let x be an arbitrary object of $\mathcal{G}$ and consider the monoid of endomorphisms* $\mathrm{End}_{\mathcal{G}}(x)$ *within $\mathcal{G}$ considered as a topological category with one object. Then the inclusion functor* $\mathrm{End}_{\mathcal{G}}(x) \longrightarrow \mathcal{G}$ *is a weak equivalence of topological categories.*

Proof. The inclusion functor is fully faithful because it induces the identity on the single mapping space. It is essentially surjective because $\pi_0\mathcal{G}$ is a connected groupoid. □

4. A model for the fundamental ∞-groupoid

The equivalence of Theorem A is realized by applying to a topological space X a certain composition of functors – which will be introduced in Section 5 – yielding an ∞-groupoid $|\mathfrak{C}(\mathrm{Sing}\,X)|$; Theorem A informally says that this ∞-groupoid models the homotopy type of X. But this object is not very transparent nor manageable without familiarity with simplicial sets. In this section we propose a model $\Pi(X)$ for the fundamental ∞-groupoid, which we believe to improve on this, being more intuitive and accessible. It will be the goal of the next section to show that $\Pi(X)$ is weakly equivalent to $|\mathfrak{C}(\mathrm{Sing}(X))|$, thus showing that it is a faithful model.

The construction of $\Pi(X)$ will be based on the following ideas. Let X be a topological space and $x \in X$ be a basepoint. Consider the space of loops ΩX based at x, that is the space of continuous paths from the unit interval into X which begin and end at x, with the compact-open topology. We can consider a binary operation on it given by concatenation of loops, which one can achieve, for example, reparametrizing the two loops to concatenate so that we obtain a new map from the unit interval.

This way of composing loops does not make the operation associative, and thus does not equip ΩX with a topological monoid structure. One way to accomplish this is by extending the space of parametrization for the paths so that we don't have to reparametrize when concatenating. This is achieved by the construction we describe next.

Consider the topological space

$$\Omega^M(X,x) := \{(f,r) \in X^{\mathbb{R}_{\geq 0}} \times \mathbb{R}_{\geq 0} \mid f(0) = f(r) = x,\ f(s) = f(r) \text{ for } s \geq r\},$$

with the product topology, where $\mathbb{R}_{\geq 0} = [0,\infty)$ has the Euclidean topology and $X^{\mathbb{R}_{\geq 0}}$ – the space of continuous maps from $\mathbb{R}_{\geq 0}$ to X – has the compact-open topology. This space is known as the space of *Moore loops*. Notice that the usual loop space ΩX includes into $\Omega^M X$ as the subspace of loops (f,r) with parameter $r = 1$. In fact, the space of Moore loops deformation retracts onto ΩX, so they are homotopy equivalent spaces.

We will now see how an associative composition may be defined on $\Omega^M X$ by looking directly at the model we propose for the fundamental ∞-groupoid.

Construction 4.1. Let X be a topological space. We let $\Pi(X)$ be the following topological category:

- The objects are the points of X.
- If x and y are two objects, we let $\mathrm{Map}_{\Pi(X)}(x, y)$ be the space of *Moore paths* from x to y:

$$\{(f, r) \in X^{\mathbb{R}_{\geq 0}} \times \mathbb{R}_{\geq 0} \mid f(0) = x,\ f(r) = y,\ f(s) = f(r) \text{ for } s \geq r\}.$$

- If g is a path that begins where f ends, we define $(g, s) \circ (f, r)$ as $(f * g, r + s)$, where

$$(f * g)(t) = \begin{cases} f(t) & \text{if } 0 \leq t \leq r, \\ g(t - r) & \text{if } t \geq r. \end{cases}$$

- The constant path $(c_x, 0)$ as identity.

Proposition 4.2. *The topological category $\Pi(X)$ is an ∞-groupoid for every space X.*

Proof. Every Moore path (f, r) has a homotopy inverse (g, r), where g goes along the same trajectory as f in reverse. □

The assignment of the fundamental ∞-groupoid to a topological space is clearly functorial, and preserves weak equivalences. Thus we have a functor $\Pi\colon \mathbf{Top} \longrightarrow \infty\text{-}\mathbf{Grpd}$ which induces a functor between homotopy categories.

Example 4.3. Let X be a topological space, and $x \in X$ a point. The space of endomorphisms of x within the ∞-groupoid $\Pi(X)$ is the space of Moore loops $\Omega^M(X, x)$ based at x. Lemma 3.4 says that the inclusion functor $\Omega^M(X, x) \longrightarrow \Pi(X)$ is a weak equivalence.

We know aim to prove that this model is accurate.

5. Realizing the fundamental ∞-groupoid

After a short technical discussion we will reach the main result of this section: Theorem B. This discussion involves a comparison with a combinatorial version of topological spaces: simplicial sets.

The combinatorial geometry of simplicial sets arises from the *simplex category*: the category whose objects are finite totally ordered sets, and whose morphisms are order-preserving functions between them. The simplex category is denoted by Δ. In it, there are certain maps, called *face* and *degeneracy* maps, which encode all the combinatorics.

Let $\mathcal{C}$ be a category. A *simplicial object* in $\mathcal{C}$ is a functor $\Delta^{\mathrm{op}} \longrightarrow \mathcal{C}$. A simplicial object in the category of sets is called a *simplicial set*. A simplicial object in the category of topological spaces is called a *simplicial space*. We denote the category of simplicial objects in $\mathcal{C}$ by $\mathbf{sC}$.

Simplicial sets have a notion of geometric realization, which is a functor to the category of topological spaces; denoted by $|-|$. For example, the geometric realization of a simplicial set X is obtained by gluing standard n-simplices, one for each element of X_n, according to the data of the face and degeneracy maps. That is, we glue the spaces $X_n \times \Delta^n$ along the mentioned maps.

More precisely, and a bit more generally, let $\mathcal{C}$ be a category with all *colimits* – in the sense of category theory, a reference for which is [6] – and suppose that we have a functor $c\colon \Delta \longrightarrow \mathcal{C}$. Then, associated to this given data, we define the geometric realization as the left Kan extension of c along the Yoneda embedding $\Delta \longrightarrow \mathbf{sSet}$.

This generalizes to simplicial spaces and bisimplicial sets – that is simplicial objects in simplicial sets – giving functors to the categories of topological spaces and of simplicial sets, respectively. For example, if X is a simplicial space, then X_n is a topological space, and we glue the product spaces $X_n \times \Delta^n$ along the face and degeneracy maps. If the X_n have the discrete topology, i.e., they are just sets, then we recover the geometric realization of simplicial sets.

Geometric realizations come with a right adjoint which we may call a *nerve* functor. For the geometric realization of simplicial sets, this functor would be the singular complex functor Sing – together with the geometric realization functor, these are the functors establishing the homotopy-theoretic equivalence between topological spaces and simplicial sets in [8] mentioned in the introduction.

As a different example, the nerve of a category $\mathcal{C}$ is a simplicial set whose set of n-simplices is the set of sequences of n composable maps. If $\mathcal{C}$ is a topological or a simplicial category, then this construction clearly generalizes and gives are simplicial space or a bisimplicial set, respectively.

Consider the composition of the nerve and realization functors

$$\mathbf{Cat_{top}} \xrightarrow{N} \mathbf{sTop} \xrightarrow{|-|} \mathbf{Top}.$$

We call this composition the *classifying space* functor, and we denote it by B. The reason behind this name is the observation that if a topological category has only one object, then it is the same as a topological monoid; if furthermore it is groupoid, then it is a topological group. For details about this construction see [10].

Lemma 5.1. *The classifying space functor B carries weak equivalences of topological categories to weak equivalences of topological spaces.*

Proof. The nerve functor clearly carries a weak equivalence of topological categories to degree-wise weak equivalences of simplicial spaces, i.e., to weak equivalences between the respective spaces of n-simplices, for every natural number n. Then one notes that the degeneracy maps of the simplicial space $N\mathcal{C}$ are closed cofibrations because points are closed, by the *weakly Hausdorff* condition in the definition of compactly generated topological space. Thus it follows from [11, A1] that the geometric realization of NF is a weak equivalence of topological spaces. □

Now, a result relating geometric realizations and nerves in the different contexts within which our main results fit. It will allow us to use the mentioned proof of Theorem A, which relies heavily on the theory of simplicial sets, to prove Theorem B.

To state the following result, we introduce the simplicial analogue of topological categories. A *simplicial category* is a category enriched over the category of simplicial sets. The category of simplicial categories is denoted by $\mathbf{Cat_\Delta}$. Theorem A is partly based on a homotopy theoretic equivalence given by a geometric realization $\mathfrak{C}$ and nerve $\mathfrak{N}$ pair of adjoint functors. This nerve is known as the *homotopy coherent nerve*, and it is a functor $\mathfrak{N}\colon \mathbf{Cat_\Delta} \longrightarrow \mathbf{sSet}$.

Proposition 5.2. *The following diagram commutes up to weak equivalence in* **Top***:*

$$\begin{tikzcd} \mathbf{Top} & \mathbf{sTop} \arrow[l, "|-|"'] & \mathbf{Cat_{top}} \arrow[l, "N"'] \arrow[ll, bend right, "B"'] \arrow[d, "\mathrm{Sing}"] \\ \mathbf{sSet} \arrow[u, "|-|"] & \mathbf{ssSet} \arrow[l, "|-|"'] \arrow[u, "|-|"] & \mathbf{Cat_{\Delta}} \arrow[l, "N"'] \arrow[ll, bend left, "\mathfrak{N}"] \end{tikzcd}$$

Proof. The left square diagram commutes up to homeomorphism – a reference for this is [9, p. 94]. For the right square diagram, we have a degree-wise weak equivalence $|N(\mathrm{Sing}\,\mathcal{G})| \longrightarrow N\mathcal{G}$, for every ∞-groupoid $\mathcal{G}$, which $|-|$ carries to a weak equivalence in **Top**, as in Lemma 5.1. The top diagram is just how we defined B. The bottom diagram commutes up to homotopy by [2, Cor. 2.6.3]. □

Proposition 5.2 says, in particular, that the diagram below at the left commutes up to weak equivalence.

$$\begin{tikzcd} \mathbf{Top} & \mathbf{Cat_{top}} \arrow[l, "B"'] \arrow[d, "\mathrm{Sing}"] \\ \mathbf{sSet} \arrow[u, "|-|"] & \mathbf{Cat_{\Delta}} \arrow[l, "\mathfrak{N}"'] \end{tikzcd} \qquad \begin{tikzcd} \mathbf{Top} \arrow[r, "\Pi"] \arrow[d, "\mathrm{Sing}"'] & \mathbf{Cat_{top}} \\ \mathbf{sSet} \arrow[r, "\mathfrak{C}"] & \mathbf{Cat_{\Delta}} \arrow[u, "|-|"'] \end{tikzcd}$$

What the next result says is that the same holds for the diagram at the right, and by Theorem A, this proves:

Theorem B. *The ∞-groupoid $\Pi(X)$ is weakly equivalent to $|\mathfrak{C}(\mathrm{Sing}(X))|$ for every topological space X.*

Proof. By Theorem A, it suffices to show that $\Pi(X)$ is weakly equivalent to $|\mathfrak{C}(\mathrm{Sing}\,X)|$. We will show in the proof of Theorem C that there is a weak equivalence $B\Omega X \simeq X \longrightarrow B\Pi(X)$ – here we choose any basepoint for X, as we don't claim naturality of the equivalence. Thus, by Proposition 5.2, we have a weak equivalence $X \longrightarrow |\mathfrak{N}(\mathrm{Sing}\,\Pi(X))|$. Applying to it the functor $|\mathfrak{C}(\mathrm{Sing}\,-)|$ we obtain that $|\mathfrak{C}(\mathrm{Sing}\,X)|$ and $\Pi(X)$ are weakly equivalent, by the equivalence of categories of Theorem A. □

This confirms to us that $\Pi(X)$ is indeed an accurate model for the fundamental ∞-groupoid.

6. An equivalence of categories

In this section we make additional assumptions on our topological spaces and ∞-groupoids, and then prove that the functors Π and B induce an equivalence between the corresponding homotopy categories. These assumptions are what we need in order to make use of the classical equivalence between connected pointed topological spaces and group-like $\mathbb{E}_1$-*spaces*.

Let $\mathbf{Top}_*$ denote the category of pointed topological spaces, i.e., with a distinguished basepoint. Similarly, let ∞-$\mathbf{Grpd}_*$ denote the category of ∞-groupoids with a distinguished object. Any model structure carries over to the pointed setting [3, Prop. 1.1.8]. In particular we can consider homotopy categories. Now let $\mathrm{Ho}(\mathbf{Top}_*^{\geq 0})$ and $\mathrm{Ho}(\infty\text{-}\mathbf{Grpd}_*^{\geq 0})$ denote the full subcategories of the corresponding homotopy categories consisting of the connected objects.

The fundamental ∞-groupoid functor Π and the classifying space functor B preserve weak equivalences. Thus they induce a pair of functors between the homotopy categories above, which is in fact an equivalence:

Theorem C. *The fundamental ∞-groupoid functor Π and the classifying space functor B induce an equivalence of categories* $\mathrm{Ho}(\mathbf{Top}_*^{\geq 0}) \simeq \mathrm{Ho}(\infty\text{-}\mathbf{Grpd}_*^{\geq 0})$.

Proof. Observe that the inclusion functor of the topological monoid $\Omega^M(X, b)$ of Moore loops, based at the basepoint b of a path-connected pointed topological space (X, b), into $\Pi(X, b)$ is a weak equivalence. Since B preserves weak equivalences by Lemma 5.1, we have a natural weak equivalence $B\Omega^M(X, b) \longrightarrow B\Pi(X, b)$ induced by the inclusion.

The homotopy equivalence $\Omega^M(X, b) \simeq \Omega(X, b)$ reduces the proof to the classical equivalence between connected pointed topological spaces and group-like $\mathbb{E}_1$*-spaces* – a reference for this is [11, Prop. 1.4]. On the one hand it gives a natural weak equivalence $(X, b) \simeq B\Omega^M(X, b) \longrightarrow B\Pi(X, b)$, by the above. On the other hand, any connected pointed ∞-groupoid $(\mathcal{G}, b)$ is weakly equivalent to the topological monoid of endomorphisms $\mathcal{M}$ at its basepoint – a weak equivalence is given by the inclusion. The property of $\mathcal{G}$ being an ∞-groupoid then says that $\pi_0\mathcal{M}$ is a group. Then the cited result says that $\mathcal{M} \simeq \Omega B\mathcal{M}$. As above, we then have natural weak equivalences

$$(\mathcal{G}, b) \longleftarrow \mathcal{M} \simeq \Omega^M B(\mathcal{G}, b) \longrightarrow \Pi B(\mathcal{G}, b)$$

induced by the inclusions. Finally, notice that these weak equivalences give natural isomorpisms in the correspondent homotopy categories. □

References

[1] A. Grothendieck, "À la poursuite des champs", Unpublished letter to Quillen (1983).

[2] V. Hinich, "Homotopy coherent nerve in Deformation theory", Preprint (2007), https://arxiv.org/abs/0704.2503.

[3] M. Hovey, *Model Categories*, Mathematical Surveys and Monographs **63**, American Mathematical Society, 2007.

[4] A. Ilias, "Model structure on the category of small topological categories", *J. Homotopy Relat. Struct.* **10**(1) (2015), 63–70.

[5] J. Lurie, *Higher Topos Theory*, Annals of Mathematics Studies, Princeton University Press, 2009.

[6] S. Mac Lane, *Categories for the Working Mathematician*, Graduate Texts in Mathematics **5**, Springer Science & Business Media, New York, 2013.

[7] J. McGarry Furriol, "Homotopical realizations of infinity groupoids", Bachelor thesis, University of Barcelona, 2020.

[8] D. Quillen, *Homotopical Algebra*, Lecture Notes in Mathematics **43**, Springer-Verlag, Berlin, Heidelberg, 1967.

[9] D. Quillen, "Higher algebraic K-theory: I", In: *Higher K-theories*, Lecture Notes in Mathematics **341**, Springer, Berlin, Heidelberg, 1973, pp. 85–147.

[10] G. Segal, "Classifying spaces and spectral sequences", *Publ. Math. Inst. Hautes Études Sci.* **34** (1968), 105–112.

[11] G. Segal, "Categories and cohomology theories", *Topology* **13**(3) (1974), 293–312.