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Hirzebruch signature theorem and exotic smooth structures on the 7-sphere

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Resum *(CAT)*
L'existència d'estructures diferencials no estàndards per a $\mathbb{S}^n$ no es va demostrar fins l'any 1956, quan J. Milnor va donar una construcció explícita d'una sèrie d'exemples pel cas $n = 7$, [4]. Fins llavors, s'assumia que no hi havia cap diferència fonamental entre esferes topològiques i esferes llises. El descobriment va suposar un punt d'inflexió en la topologia algebraica i de varietats, que continuaria amb la caracterització dels anomenats grups d'esferes homotòpiques, [2]. Un dels resultats que va fer possible la prova de Milnor va ser el teorema de la signatura de Hirzebruch, que dona una fórmula pel càlcul de la signatura d'una varietat (diferenciable) compacta i orientada. L'objectiu d'aquest treball és contextualitzar aquest teorema, així com mostrar el seu paper en la construcció de les primeres 7-esferes exòtiques.

Abstract *(ENG)*
The existence of non-standard smooth structures on $\mathbb{S}^n$ was not proven until 1956, when J. Milnor presented an explicit construction for the case $n = 7$, [4]. Until then, it was assumed that there was no fundamental difference between topological and smooth spheres. This had profound implications in the field of manifold and algebraic topology, and was immediately endorsed by subsequent research, which lead to the characterization of the so called groups of homotopy spheres, [2]. One of the results that made Milnor's approach possible was Hirzebruch's signature theorem, which gives a formula to compute the signature of a (smooth) compact oriented manifold. The aim of this work is to contextualize this theorem, as well as to show its role in the construction of the first exotic 7-spheres.

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1. Classification of vector bundles

An *n-real vector bundle* is a triplet (π, E, B), where E (*total space*) and B (*base space*) are topological spaces and $\pi\colon E \longrightarrow B$ (*projection*) is a continuous map s.t. for each $b \in B$, $\pi^{-1}(b) \cong \mathbb{R}^n$ as real vector spaces, and there exists a neighbourhood U containing b s.t. $U \times \mathbb{R}^n \cong \pi^{-1}(U)$ through a homeomorphism h that restricts to a linear isomorphism $\{b\} \times \mathbb{R}^n \longrightarrow \pi^{-1}(b)$ for each $b \in U$. We refer to the pair (U, h) as a *local coordinate system for* ξ and we say that ξ is *locally trivial*.

The notion of an induced bundle along a map is ubiquitous in the constructions that are to follow and will be used extensively throughout the work. Given a vector bundle $\xi := (\pi, E, B)$, a topological space B' and a continuous map $f\colon B' \longrightarrow B$, one can define another vector bundle, $f^*\xi := (\pi', E', B')$, with $E' := \{(b, e) \in B' \times E \mid \pi(e) = f(b)\}$ and $\pi'\colon E' \longrightarrow B'$, $(b, e) \longmapsto b$. The vector space structure in the fibers, $(\pi')^{-1}(b)$, is defined by $\lambda_1(b, e_1) + \lambda_2(b, e_2) = (b, \lambda_1 e_1 + \lambda e_2)$. We refer to $f^*\xi$ as the *induced bundle* or *pullback bundle* of ξ by f. One can show that $f^*\xi$ is locally trivial by expressing its local coordinate systems in terms of local coordinate systems for ξ. The reader is referred to [5, §3] for a more detailed description.

One of the most important results regarding real vector bundles is the following classification theorem (see [5, §2–5], [1, §1]), which translates the problem of classifying isomorphic vector bundles into a homotopy problem:

Theorem 1.1. *Let $\mathfrak{F}_n(B)$ be the set of n-vector bundles over a paracompact Hausdorff base B modulo isomorphism, and let $[B, G_n]$ be the set of homotopy classes of maps from B to G_n. Then, the map:*

$$\begin{aligned} \Phi\colon [B, G_n] &\longrightarrow \mathfrak{F}_n(B) \\ [f] &\longmapsto [f^*\gamma^n] \end{aligned} \tag{1}$$

is a bijection; where γ^n is the universal bundle over $G_n(\mathbb{R}^\infty)$.

To prove this theorem we need three important results. We will explain each of them and briefly detail their proofs. The first one is necessary to ensure that Φ is well defined:

Proposition 1.2. *Let ξ be a vector bundle with projection $\pi\colon E \longrightarrow B$, and let $f, g\colon C \longrightarrow B$ be continuous maps, with C paracompact*[1]*. If f and g are homotopic, then $f^*\xi \cong g^*\xi$.*

Sketch of proof. The proof of Proposition 1.2 is based on the fact that every n-vector bundle over the interval $[0, 1] =: I$ is trivial. From this, one then shows that, given a vector bundle on C, there is an open cover such that for each of its open sets U_i the restriction of the bundle to $U_i \times I$ is trivial. In particular, on each of these open sets, the bundles at each end are clearly isomorphic. One then glues these local isomorphisms to show that, given $i_0, i_1\colon C \longrightarrow C \times I$, $i_k(c) = (c, k)$, $k = 0, 1$, and a vector bundle ξ over $C \times I$, the induced bundles by i_0 and i_1 over C are isomorphic, which allows us to complete de proof. Indeed, given $f, g\colon C \longrightarrow B$ homotopic, and $h\colon C \times I \longrightarrow B$ a homotopy s.t. $h_0 = f$, $h_1 = g$, we have $f = h \circ i_0$, $g = h \circ i_1$, which implies $f^*\xi \cong i_0^*(h^*\xi)$, $g^*\xi \cong i_1^*(h^*\xi)$. This yields $f^*\xi \cong g^*\xi$, by the observation on the induced bundles by i_0 and i_1. □

[1] Paracompactness (every open cover admits a locally finite refinement) will be necessary to ensure we can use partitions of unity arguments.

The second of the aforementioned results is necessary to ensure that Φ is surjective:

Proposition 1.3. *Any n-vector bundle over a paracompact Hausdorff base B admits a bundle morphism $f\colon \xi \longrightarrow \gamma^n$.*

Sketch of proof. This result is an extension of its "finite" counterpart, which ensures the existence of a bundle morphism $\xi \longrightarrow \gamma_k^n$ (for a sufficiently large k), with γ_k^n the canonical vector bundle over the Grassmannian $G_n(\mathbb{R}^{n+k})$ (following the notation in [5]). The latter assumes B compact and Hausdorff, which allows us to take a finite covering $\{U_1, \dots, U_r\}$ s.t. $\xi \mid U_i$ is trivial for each i. Thus, each U_i admits n linearly independent sections. By using partitions of unity, one can extend these sections over B, and find a finite number of them, $s_1, \dots, s_n, \dots, s_{n+k}$, s.t. for every $b \in B$, $\{s_1(b), \dots, s_{n+k}(b)\}$ generates $F_b(\xi)$. Thus, the map $g_b\colon (t_1, \dots, t_{n+k}) \longmapsto \Sigma t_i s_i(b)$ between $\mathbb{R}^{n+k}$ and $F_b(\xi)$ is surjective for each b. Denoting $V_b := (\ker g_b)^\perp$ and f_b the linear isomorphism given by $g_b \mid V_b$, it is clear that the map $f\colon e \longmapsto (V_b, f_b^{-1}(e))$ defines an isomorphism between ξ and γ_k^n. The proof for the γ^n case works similarly, letting k go to infinity and taking into account that the V_b's are embedded in $\mathbb{R}^\infty$, which allows to weaken the condition on the base, B (hence the paracompactness). □

The last of these three core results related to Theorem 1.1 proves that Φ is injective:

Proposition 1.4. *Let ξ be an n-vector bundle over a paracompact Hausdorff base B, and let $f, g\colon \xi \longrightarrow \gamma^n$ be bundle morphisms. Then, f and g are homotopic.*

Sketch of proof. From the proof of Proposition 1.3 it is clear that any morphism $f\colon \xi \longrightarrow \gamma^n$ is of the form $e \longmapsto (\tilde{f}$ (fiber over e)$, \tilde{f}(e))$, where $\tilde{f}$ is a continuous map between $E(\xi)$ and $\mathbb{R}^\infty$ that is linear injective over the fibers of ξ. Let $\tilde{f}, \tilde{g}$ be these maps for f and g respectively. Two scenarios are distinguished, based on the relation between $\tilde{f}$ and $\tilde{g}$. Firstly, we assume $\tilde{f}(e) \neq \lambda \tilde{g}(e)$ for any $\lambda < 0$, with $e \in E(\xi)$. Then, we can easily define a bundle homotopy $h\colon E(\xi) \times [0,1] \longrightarrow E(\gamma^n)$ between f and g of the form $(e,t) \longmapsto (\tilde{h}_t$ (fiber over e)$, \tilde{h}_t(e))$, by setting $\tilde{h}_t(e) := (1-t)\tilde{f}(e) + t\tilde{g}(e)$. The condition on $\tilde{f}$ and $\tilde{g}$ allows to prove the injectivity of $\tilde{h}_t$, and the fact that f and g are both bundle maps, ensures continuity and linearity, which proves h_t is a morphism for each t. Moreover, it can also be seen that h is continuous, by inspecting its restriction to the bases. This proves h is a homotopy. Secondly, we make no assumption on $\tilde{f}$ and $\tilde{g}$. To prove that f and g are homotopic, we consider the maps $\tilde{d}_1, \tilde{d}_2\colon \mathbb{R}^\infty \longrightarrow \mathbb{R}^\infty$ s.t. $\tilde{d}_1(e_i) = e_{2i-1}$, $\tilde{d}_2(e_i) = e_{2i}$, for $i = 1, 2, 3, \dots$, where the e_j's are the canonical base vectors for $\mathbb{R}^\infty$. These maps induce two morphisms d_1, d_2 from γ^n to itself, that we can use to obtain $r_1 := d_1 \circ f$ and $r_2 := d_2 \circ g$. It is clear that $\tilde{f}(e) \neq \lambda \tilde{r}_1(e)$, with $\lambda < 0$, for all $e \in E(\xi)$, which implies $f \simeq r_1$. Analogously, $g \simeq r_2$. We can verify that the same condition holds between $\tilde{r}_1$ and $\tilde{r}_2$. Thus, $f \simeq g$, as desired. □

With these results, we can prove Theorem 1.1:

Proof of Theorem 1.1. Proposition 1.2 guarantees that Φ is a well defined map between $[B, G_n]$ and $\mathfrak{F}_n(B)$. We can see that Φ is injective using Proposition 1.4. Indeed, $[f^*\gamma^n] = [g^*\gamma^n]$ implies that there exists an isomorphism $\phi\colon E(f^*\gamma^n) \longrightarrow E(g^*\gamma^n)$. Composing with $\hat{g}\colon E(g^*\gamma^n) \longrightarrow E(\gamma^n)$ we obtain a bundle morphism $\varphi := \hat{g} \circ \phi$ with induced base map $\overline{\varphi} = g \circ \overline{\phi} = g$, given that $\overline{\phi} = \mathrm{id}$, since ϕ is an isomorphism. By Proposition 1.4, φ and $\hat{f}$ are homotopic bundle morphisms, which yields $[\overline{\varphi}] = [g] = [f]$, as desired.

We now prove that Φ is surjective. Let $[\xi] \in \mathfrak{F}_n(B)$. By Proposition 1.3 we know there exists a bundle morphism $f\colon \xi \longrightarrow \gamma^n$ of the form $f(e) = (\tilde{f}$ (fiber over e)$, \tilde{f}(e))$, with $\tilde{f}\colon E(\xi) \longrightarrow \mathbb{R}^\infty$ continuous and linear injective at the level of the fibers. This provides us with a homotopy class $[\overline{f}]$ where $\overline{f}$ is the base induced map associated to f, *i.e.* $b \longmapsto \tilde{f}(F_b(\xi))$. We observe that $\overline{f}^*\gamma^n$ and ξ are isomorphic, since f is a morphism. Thus, $\Phi([\overline{f}]) = [\xi]$, as desired. □

An *n-complex vector bundle* ω is defined similarly, with all $\mathbb{R}$-linear objects and properties substituted by their $\mathbb{C}$-linear analogues. We also have a classification theorem for these bundles, similar to Theorem 1.1. However, complex vector bundles present a much richer structure than their real counterparts. On one hand, every complex vector bundle can be thought as a real vector bundle, by omitting its *complex structure* (we can just think of its fibers as real vector spaces of twice the original dimension). We refer to this bundle as the *underlying real vector bundle of* ω, and denote it by $\omega_{\mathbb{R}}$. On the other hand, we can define the *conjugate bundle of* ω, $\overline{\omega}$, as the complex vector bundle with same underlying bundle, *i.e.* $\overline{\omega}_{\mathbb{R}} = \omega_{\mathbb{R}}$, but with opposite complex structure, *i.e.* $\mathrm{id}\colon E(\omega) \longrightarrow E(\overline{\omega})$ is $\mathbb{C}$-conjugate linear on fibers.

2. Characteristic classes

Theorem 1.1 is key to define *characteristic classes* for vector bundles. Let c be a cohomology class in $H^i(G_n; R)$, with R some coefficient ring, and let ξ be a certain n-vector bundle with (paracompact Hausdorff) base B. From Theorem 1.1 there is a unique homotopy class $[\overline{f}_\xi] \in [B, G_n]$ s.t. $\overline{f}^*{}_\xi\gamma^n \cong \xi$. Let $\overline{f}^*{}_\xi\colon H^i(G_n; R) \longrightarrow H^i(B; R)$ be the induced morphism between cohomology groups, and define[2] $c(\xi) := \overline{f}^*{}_\xi c$. This cohomology class in $H^i(B; R)$ is the *characteristic class of* ξ *determined by* c. From this construction we make two important observations. First, the correspondence $\xi \longmapsto c(\xi)$ is *natural* w.r.t. bundle morphisms, meaning that if we have $g\colon B(\xi) \longrightarrow B(\eta)$ covered by a bundle morphism, then $c(\xi) = g^*c(\eta)$. This can be seen by noting that $\overline{f}_\eta \circ g$ and $\overline{f}_\xi$ are homotopic. Since g is covered by a bundle morphism, $\xi \cong g^*\eta$, which allows us to write $c(g^*\eta) = g^*c(\eta)$. Second, we note that given any "natural" correspondence $\xi \longmapsto c(\xi)$, we will necessarily have $c(\xi) = \overline{f}^*{}_\xi c(\gamma^n)$, since the map $\overline{f}_\xi$ is always covered by a bundle morphism between ξ and γ^n. So this construction is as general as one can ask for and tells us that the ring of characteristic cohomology classes of n-vector bundles over paracompact Hausdorff base, with coefficients in R, is canonically isomorphic to $H^*(G_n; R)$. Thus, the computation of $H^*(G_n; R)$ is relevant to the study of fundamental properties of n-vector bundles over paracompact Hausdorff base. We will briefly present the most important 3 types of characteristic classes: the Stiefel–Whitney classes, the Chern classes and the Pontrjagin classes.

From now on, we assume that all vector bundles have paracompact Hausdorff bases, unless otherwise stated. The *Stiefel–Whitney classes* are characteristic classes of non oriented vector bundles, that are defined in $\mathbb{Z}/2$ cohomology groups. The existence and uniqueness of such classes is difficult to prove, and is based on the computation of $H^*(G_n; \mathbb{Z}/2)$, which can be found, for example, in [5, §6–8]. Hence, most often we find these classes defined axiomatically as follows:

Definition 2.1 (Stiefel–Whitney classes. Axiomatic definition)**.**

AXIOM I. Given a vector bundle ξ, there is a unique sequence of characteristic classes $w_i(\xi) \in H^i(B(\xi); \mathbb{Z}/2)$, $i = 0, 1, 2, \dots$, that we refer to as *the Stiefel–Whitney classes of* ξ. Moreover, $w_0(\xi) = 1$ and if ξ is an n-vector bundle, $w_i(\xi) = 0$ for all $i > n$.

[2] Note that this is well defined, because the induced morphisms from two maps sharing homotopy type are equal.

Axiom II. Naturality If $f\colon B(\xi) \longrightarrow B(\eta)$ is covered by a bundle morphism between ξ and η, then: $w_i(\xi) = f^* w_i(\eta)$, $i = 0, 1, 2\dots$, where f^* is the induced morphism between cohomology groups over $\mathbb{Z}/2$.

Axiom III. Let ξ and η be two vector bundles over the same base, B, then:

$$w_k(\xi \oplus \eta) = \sum_{i=0}^{k} w_i(\xi) \smile w_{k-i}(\eta), \quad k = 0, 1, 2, \dots, \tag{2}$$

where $\smile$ denotes the *cup product* between cohomology classes.

Axiom IV. $w_1(\gamma_1^1) \neq 0$, where γ_1^1 is the canonical line bundle over $\mathbb{P}^1$.

We define the *total Stiefel–Whitney class* $w(\xi)$, for an n-vector bundle ξ, as the formal sum $w_0(\xi) + w_1(\xi) + \cdots + w_n(\xi) + 0 + \cdots$ in the ring $H^{\Pi}(B(\xi); \mathbb{Z}/2)$ of series of the form $a_0 + a_1 + a_2 + \cdots$, with each $a_i \in H^i(B(\xi); \mathbb{Z}/2)$. This notation serves to express more synthetically Axiom III from Definition 2.1, which can now be written as $w(\xi \oplus \eta) = w(\xi) w(\eta)$.

Some basic properties of the Stiefel–Whitney classes are:

Proposition 2.2.

(i) *The Stiefel–Whitney classes of two isomorphic bundles are equal.*

(ii) *Let ε be the trivial bundle, then $w_i(\varepsilon) = 0$ for $i > 0$.*

(iii) *Let η be a vector bundle and ε the trivial bundle over the same base, then $w_i(\varepsilon \oplus \eta) = w_i(\eta)$ for $i = 0, 1, 2, \dots$*

(iv) *Let ξ be an euclidian n-vector bundle with a non-zero section $s\colon B \longrightarrow E$; then $w_n(\xi) = 0$.*
More generally, if ξ has k linearly independent sections, then $w_{n-k+1}(\xi) = \cdots = w_n(\xi) = 0$.

From the axioms in Definition 2.1 we can compute the total Stiefel–Whitney class of the canonical line bundle over the real projective space $\mathbb{P}^n$, γ_n^1:

Proposition 2.3. *$w(\gamma_n^1) = 1 + a$, where a is the generator of $H^1(\mathbb{P}^n; \mathbb{Z}/2)$.*

Proof. Let $j\colon \mathbb{P}^1 \longrightarrow \mathbb{P}^n$ be the inclusion $[x] \longmapsto [i(x)]$, with i the canonical inclusion $\mathbb{R}^2 \hookrightarrow \mathbb{R}^{n+1}$. The map j is covered by the bundle morphism J between γ_1^1 and γ_n^1 that sends $([x], v) \in E(\gamma_1^1)$ to $(j([x]), i(v)) \in E(\gamma_n^1)$. By Axioms II and IV, we then have $j^* w_i(\gamma_n^1) = w_i(\gamma_1^1)$ and $j^* w_1(\gamma_n^1) \neq 0$, respectively. Since j^* is a morphism, $w_1(\gamma_n^1) \neq 0$ necessarily, and by the structure of the cohomology ring $H^*(\mathbb{P}^n; \mathbb{Z}/2)$, $w_1(\gamma_n^1) = a$, with a its generator. Finally, since $\dim \gamma_n^1 = 1$, $w_i(\gamma_n^1) = 0$ for $i > 1$ and $w_0(\gamma_n^1) = 1$, by Axiom I. Thus, $w(\gamma_n^1) = 1 + a$, which concludes the proof. □

Proposition 2.4. *The total Stiefel–Whitney class of $T\mathbb{P}^n$ is $(1 + a)^{n+1}$, where $a \in H^1(\mathbb{P}^n; \mathbb{Z}/2)$ is the ring generator of $H^*(\mathbb{P}^n; \mathbb{Z}/2)$.*

Proof. Assume that $T\mathbb{P}^n$ is isomorphic to $\mathrm{Hom}(\gamma_n^1, \gamma^\perp)$, where $\gamma^\perp$ is the orthogonal complement of γ_n^1 in $\varepsilon_{\mathbb{P}^n}^{n+1}$. Note then that $\mathrm{Hom}(\gamma_n^1, \gamma_n^1)$ is a trivial bundle of dimension 1, which implies:

$$\begin{aligned} T\mathbb{P}^n \oplus \varepsilon^1 &\cong \mathrm{Hom}(\gamma_n^1, \gamma^\perp) \oplus \mathrm{Hom}(\gamma_n^1, \gamma_n^1) \cong \mathrm{Hom}(\gamma_n^1, \gamma_n^1 \oplus \gamma^\perp) \\ &\cong \mathrm{Hom}(\gamma_n^1, \varepsilon^{n+1}) \cong \mathrm{Hom}(\gamma_n^1, \varepsilon^1 \oplus \cdots \oplus \varepsilon^1) \\ &\cong \mathrm{Hom}(\gamma_n^1, \varepsilon^1) \oplus \cdots \oplus \mathrm{Hom}(\gamma_n^1, \varepsilon^1) \cong \gamma_n^1 \oplus \cdots \oplus \gamma_n^1, \end{aligned} \tag{3}$$

where we used $\mathrm{Hom}(\gamma_n^1, \varepsilon^1) \cong \gamma_n^1$ and $\gamma_n^1 \oplus \gamma^\perp \cong \varepsilon^{n+1}$, by construction. Thus, using property (iii) from Proposition 2.2, Axiom III and Proposition 2.3 it is clear that:

$$w(T\mathbb{P}^n) = w(T\mathbb{P}^n \oplus \varepsilon^1) = w(\gamma_n^1 \oplus \cdots \oplus \gamma_n^1) = w(\gamma_n^1)^{n+1} = (1+a)^{n+1}. \tag{4}$$

We now prove $T\mathbb{P}^n \cong \mathrm{Hom}(\gamma_n^1, \gamma^\perp)$. Let $q\colon \mathbb{S}^n \longrightarrow \mathbb{P}^n$ be the quotient map. Note that $T_x q(v) = T_{-x} q(-v)$, which can be derived from the fact that $q(x) = q(-x)$. Since q is a local diffeomorphism, $T_x q$ is a linear isomorphism for each x. This allows identifying $T\mathbb{P}^n$ with the pairs $\{(x,v),(-x,-v)\}$ s.t. $\|x\| = 1$ and $\langle x, v\rangle = 0$ through $\{(x,v),(-x,-v)\} \longmapsto ([x], T_x q(v))$. Observe that each of these pairs defines a linear map from $\langle x\rangle = F_{[x]}(\gamma_n^1)$ to $\langle x\rangle^\perp = F_{[x]}(\gamma^\perp)$, determined by $x \longmapsto v$. We can identify this map with the corresponding element in $F_{[x]}(\mathrm{Hom}(\gamma_n^1, \gamma^\perp))$. This clearly allows defining a bundle isomorphism between $T\mathbb{P}^n$ and $\mathrm{Hom}(\gamma_n^1, \gamma^\perp)$:

$$\begin{gathered} \varphi\colon T\mathbb{P}^n \longrightarrow E(\mathrm{Hom}(\gamma_n^1, \gamma^\perp)) \\ ([y], u) \sim \{(x,v),(-x,-v)\} \longmapsto L_{[y],u}\colon F_{[y]}(\gamma_n^1) \longrightarrow F_{[y]}(\gamma^\perp),\ x \longmapsto v \end{gathered} \tag{5}$$

with $x = \frac{y}{\|y\|}$ and $v = T_x q^{-1}(u)$. □

Chern classes are characteristic classes associated to complex vector bundles. They are constructed as in the real case, but using the corresponding classification theorem, and are cohomology classes over $\mathbb{Z}$. We provide, as for the Stiefel–Whitney classes, an axiomatic definition:

Definition 2.5 (Chern classes. Axiomatic definition)**.**

Axiom I. Given a complex vector bundle ω, there is a unique sequence of characteristic classes $c_i(\omega) \in H^{2i}(B(\xi);\mathbb{Z})$, $i = 0,1,2,\dots$, that we refer to as the *Chern classes of* ω. Moreover, $c_0(\omega) = 1$ and if ω is an n-complex vector bundle, $c_i(\omega) = 0$ for all $i > n$.

Axiom II. Naturality. If $f\colon B(\omega) \longrightarrow B(\omega')$ is covered by a bundle morphism between ω and ω', then: $c_i(\omega) = f^* c_i(\omega')$, $i = 0,1,2\dots$

Axiom III. Let ω and ω' be two complex vector bundles over the same base, B, then:

$$c_k(\omega \oplus \omega') = \sum_{i=0}^{k} c_i(\omega) \smile c_{k-i}(\omega'), \quad k = 0,1,2,\dots \tag{6}$$

Axiom IV. $c_1(\gamma_1^1) \neq 0$, where γ_1^1 is the canonical line bundle over $\mathbb{CP}^1$.

A *total Chern class* $c(\omega)$ is defined for complex vector bundles in the same way as in the Stiefel–Whitney case. Chern classes share properties (i)–(iii) from Proposition 2.2 with the Stiefel–Whitney classes. However, there is a distinctive feature that we need to present for future developments:

Proposition 2.6. *Given a complex vector bundle* ω, $c_k(\overline{\omega}) = (-1)^k c_k(\omega)$, $k \geq 0$, *where* $\overline{\omega}$ *is the conjugate bundle of* ω.

Proposition 2.7. *The total Chern class of* $T\mathbb{CP}^n$ *is* $(1+t)^{n+1}$, *where* $t = -c_1(\gamma_n^1)$, *and* γ_n^1 *is the canonical line bundle over* $\mathbb{CP}^n$.

Sketch of proof. We can easily follow the steps in Proposition 2.4, to show that:

$$T\mathbb{CP}^n \oplus \varepsilon^1 \cong \mathrm{Hom}(\gamma_n^1, \varepsilon^1) \oplus \overset{n+1)}{\cdots} \oplus \mathrm{Hom}(\gamma_n^1, \varepsilon^1) \cong \overline{\gamma}_n^1 \oplus \overset{n+1)}{\cdots} \oplus \overline{\gamma}_n^1, \tag{7}$$

where we have used that the dual of a complex vector bundle equipped with an hermitian metric is canonically isomorphic to its conjugate bundle. This can be seen by noting that the map $v \longmapsto \langle \cdot, v \rangle$ between $E(\overline{\omega})$ and $E(\mathrm{Hom}(\omega, \varepsilon^1))$ is a (complex) bundle isomorphism. Thus, using Proposition 2.6:

$$c(T\mathbb{CP}^n) = c(\overline{\gamma}_n^1)^{n+1} = (1 - c_1(\gamma_n^1))^{n+1} = (1+t)^{n+1}. \tag{8}$$

□

Pontrjagin classes are characteristic cohomology classes over $\mathbb{Z}$ for real, possibly oriented, vector bundles. They can be thought as the oriented analogues of the Stiefel–Whitney classes, as they allow to distinguish between different vector bundle orientations. They are defined through Chern classes as follows:

$$p_i(\xi) := (-1)^i c_{2i}(\xi \otimes \mathbb{C}) \in H^{4i}(B(\xi); \mathbb{Z}), \quad i = 0, 1, 2, \dots, \tag{9}$$

where ξ is a real vector bundle of dimension n, and $\xi \otimes \mathbb{C}$ its *complexification*, that is, the complex vector bundle over the same base whose fibers are the products $F_b(\xi) \otimes_{\mathbb{R}} \mathbb{C}$ (treating $\mathbb{C}$ as a vector space over $\mathbb{R}$). By Axiom I of the Chern classes, it is clear that $p_i(\xi) = 0$ for all $i > n/2$. Hence, the *total Pontrjagin class of* ξ can be written as: $p(\xi) := 1 + p_1(\xi) + p_2(\xi) + \cdots + p_{\lfloor \frac{n}{2} \rfloor}(\xi)$.

As they are derived from Chern classes, Pontrjagin classes also share properties (i)–(iii) from Proposition 2.2. However, they exhibit two more properties:

Proposition 2.8.

(i) Given two real vector bundles ξ, η, $p(\xi \oplus \eta) = p(\xi)p(\eta)$ modulo order 2 terms.

(ii) Let ω be an n-complex vector bundle. Then, the Chern classes of ω determine the Pontrjagin classes of $\omega_{\mathbb{R}}$, through the following relation:

$$p_k(\omega_{\mathbb{R}}) = c_k(\omega)^2 - 2c_{k-1}(\omega)c_{k+1}(\omega) + \cdots + (-1)^k 2c_{2k}(\omega). \tag{10}$$

Alternatively, we can write:

$$1 - p_1(\omega_{\mathbb{R}}) + p_2(\omega_{\mathbb{R}}) - \cdots + (-1)^n p_n(\omega_{\mathbb{R}}) = c(\omega)c(\overline{\omega}) \text{ modulo order 2 terms.} \tag{11}$$

Proposition 2.8 is, essentially, a consequence of Proposition 2.6 combined with the following bundle isomorphisms: $\xi \otimes \mathbb{C} \cong \overline{\xi \otimes \mathbb{C}}$, with ξ a real vector bundle (property (i)); and $\omega_{\mathbb{R}} \otimes \mathbb{C} \cong \omega \oplus \overline{\omega}$, with ω an n-complex vector bundle (property (ii)).

Proposition 2.9. *The total Pontrjagin class of the underlying vector bundle of $T\mathbb{CP}^n$ is $1 + \binom{n+1}{1}t^2 + \binom{n+1}{2}t^4 + \cdots + \binom{n+1}{\lfloor \frac{n}{2} \rfloor} t^{2\lfloor \frac{n}{2} \rfloor}$, where $t = -c_1(\gamma_n^1)$. Alternatively, $p(T\mathbb{CP}^n_{\mathbb{R}}) = (1+t^2)^{n+1}$.*

Proof. Denote $\tau := T\mathbb{CP}^n$. From Propositions 2.7 and 2.8(ii), it is clear that:

$$c(\tau)c(\overline{\tau}) = (1+t)^{n+1}(1-t)^{n+1} = (1-t^2)^{n+1} = 1 - p_1(\tau_{\mathbb{R}}) + p_2(\tau_{\mathbb{R}}) - \cdots + (-1)^n p_n(\tau_{\mathbb{R}}) \tag{12}$$

modulo order 2 terms. Taking into account that $t \in H^2(\mathbb{CP}^n; \mathbb{Z})$ and comparing elements with same dimension in cohomology, we find $p_k(\tau_{\mathbb{R}}) = \binom{n+1}{k} t^{2k}$, $k = 0, 1, \dots, n$, which allows us to write $p(\tau_{\mathbb{R}}) = (1+t^2)^{n+1}$. Finally, using $H^i(\mathbb{CP}^n; \mathbb{Z}) \cong 0$ for $i > 2n$, we obtain the result. □

3. Hirzebruch signature theorem

Manifold cobordism is a well known equivalence relation between closed differentiable manifolds. Given two closed differentiable n-manifolds (n-manifolds from now on) M_1, M_2, we say they belong to the same *cobordism class*, or that they are *cobordant*, and denote it $\overline{M_1} = \overline{M_2}$, if $M_1 \sqcup M_2$ is the boundary of a compact $(n+1)$-manifold, N, that we refer to as a *cobordism between* M_1 *and* M_2. Similarly, two closed n-manifolds belong to the same *oriented cobordism class* if $M_1 \sqcup (-M_2)$ is the boundary of a compact oriented $(n+1)$-manifold (through an orientation preserving diffeomorphism). These are equivalence relations (see [5, 3]) over closed manifolds and closed oriented manifolds respectively. We can provide a brief explanation to justify why this may be true for the oriented case: M is cobordant to itself, because $M \sqcup (-M) = \partial(M \times [0,1])$; if W is a cobordism between M_1 and M_2, it is clear that $-W$ is a cobordism between M_2 and M_1; finally, given W_1, W_2 cobordisms between M_1, N and N, M_2 respectively, we have that $W_1 \sqcup W_2/\sim$, conveniently identifying N with $-N$ (using the *collar neighborhood theorem*), is a cobordism between M_1 and M_2.

One of the major contributors to cobordism theory was R. Thom, who helped establishing its foundations, but also gave birth to some of its most important results. The classification of closed (oriented) manifolds up to cobordism was a consequence of the efforts by R. Thom, L. Pontrjagin and C. T. C. Wall, who were able to connect characteristic classes with cobordism classes in a brilliant manner. To present this result, we need to introduce the notion of characteristic numbers:

Definition 3.1 (Stiefel–Whitney numbers and Pontrjagin numbers)**.** Let M be a closed m-manifold and N a closed oriented $4n$-manifold. Let $I = (i_1, \dots, i_r)$ and $J = (j_1, \dots, j_s)$ be partitions of m and n respectively, and define:

$$w_I[M] := \langle w_{i_1}(TM) \cdots w_{i_r}(TM), \mu \rangle, \quad p_J[N] := \langle p_{j_1}(TN) \cdots p_{j_s}(TN), \nu \rangle, \tag{13}$$

where $\mu \in H_m(M; \mathbb{Z}/2)$ and $\nu \in H_{4n}(N; \mathbb{Z})$ are the fundamental homology classes of M and N. The numbers $w_I[M] \in \mathbb{Z}/2$ and $p_I[N] \in \mathbb{Z}$ are the Stiefel–Whitney numbers of M and the Pontrjagin numbers of N, respectively. If $\dim N \not\equiv 0 \ (4)$, we say that the Pontrjagin numbers of N vanish.

Theorem 3.2 (Classification of closed oriented manifolds modulo oriented cobordism)**.** *Stiefel–Whitney and Pontrjagin numbers completely classify closed oriented manifolds modulo oriented cobordism. Thus, given two closed oriented n-manifolds M_1 and M_2, they belong to the same oriented cobordism class if and only if $w_I[M_1] = w_I[M_2]$ and $p_J[M_1] = p_J[M_2]$ for all partitions I, J of n and $\frac{n}{4}$, respectively.*

Both oriented and unoriented cobordisms give rise to a group structure between cobordism classes of closed manifolds of the same dimension, by means of the disjoint union. Since, we are most interested in the oriented case, we present these groups and the graded ring they form, $\Omega_\star^{\mathrm{SO}}$, which is an important object in the subsequent discussion.

Definition 3.3 (Oriented cobordism groups and oriented cobordism ring)**.** We define the *oriented cobordism group of dimension n* as the set $\Omega_n^{\mathrm{SO}} := \{\overline{M} \mid M \text{ closed oriented } n\text{-manifold}\}$ together with the following operation: $\overline{M} + \overline{N} := \overline{M \sqcup N}$. $(\Omega_n^{\mathrm{SO}}, +)$ is an abelian group with identity $\overline{\varnothing}$.

We define the *oriented cobordism ring* as the graded commutative ring $\Omega_\star^{\mathrm{SO}} := \bigoplus_{n=0}^{\infty} \Omega_n^{\mathrm{SO}}$, with component-wise sum and product given by the cartesian product between manifolds: $\overline{M_1} \times \overline{M_2} := \overline{M_1 \times M_2}$.

To prove that the sum of oriented cobordism classes is well defined one must see that if $W = W_1 \sqcup W_2$, with W_1, W_1, W compact oriented $(n+1)$-manifolds, then $\partial W = \partial W_1 \sqcup \partial W_2$. On the other hand, given a cobordism W between M_1, M_2, and given a closed oriented manifold N, $W \times N$ is a cobordism between $M_1 \times N$ and $M_2 \times N$. Thus, with $\overline{M_i} = \overline{N_i}$, $i = 1, 2$, $\overline{M_1 \times M_2} = \overline{N_1 \times M_2} = \overline{N_1 \times N_2}$, which proves the product is also well defined. Note also that $\Omega_\star^{\mathrm{SO}}$ is commutative in the graded sense, because $M_1 \times M_2$ and $(-1)^{\dim M_1 \dim M_2} M_2 \times M_1$ are diffeomorphic as oriented manifolds. The structure of $\Omega_\star^{\mathrm{SO}}$ was thoroughly studied by R. Thom, whose findings can be summarized as follows (see [5, §16–17]):

Theorem 3.4.

(i) Ω_n^{SO} *is a finite group for* $n \equiv 0$ (4) *and is finitely generated with rank* $p(k)$*, the number of partitions of* k*, when* $n = 4k$*. Moreover, in the case where* $n = 4k$*, the products* $\mathbb{CP}^{2i_1} \times \cdots \times \mathbb{CP}^{2i_r}$*, with* $I = (i_1, \dots, i_r)$ *partition of* k*, are a set of independent generators.*

(ii) $\Omega_\star^{\mathrm{SO}} \otimes \mathbb{Q}$ *is a polynomial ring over* $\mathbb{Q}$ *generated by* $\mathbb{CP}^2, \mathbb{CP}^4, \mathbb{CP}^6, \dots$

Note that (ii) is a direct consequence of (i). Indeed, since the $\mathbb{Z}$-module product $\Omega_\star^{\mathrm{SO}} \otimes \mathbb{Q}$ is effectively eliminating the torsion elements in $\Omega_\star^{\mathrm{SO}}$, we are left, by (i), with the groups $\Omega_{4k}^{\mathrm{SO}}$, $k \geq 0$; given that these are generated by the products $\mathbb{CP}^{2i_1} \times \cdots \times \mathbb{CP}^{2i_r}$, the results follows. Note also that in $\Omega_\star^{\mathrm{SO}} \otimes \mathbb{Q}$ all products are commutative, given that $\dim \mathbb{CP}^{2i} \equiv 0$ (2).

An important cobordism invariant, other than Pontrjagin and Stiefel–Whitney numbers, is the signature. In fact, this section is devoted to Hirzebruch's signature theorem, which presents a formula for the computation of the signature of a manifold, in terms of its Pontrjagin numbers.

Definition 3.5 (Signature)**.** The *signature* of a compact oriented manifold M of dimension $4k$, is the signature of the rational bilinear symmetric form:

$$\begin{aligned} H^{2k}(M;\mathbb{Q}) \times H^{2k}(M;\mathbb{Q}) &\longrightarrow \mathbb{Q} \\ (u, v) &\longmapsto \langle u \smile v, \mu \rangle, \end{aligned} \tag{14}$$

where μ is the fundamental homology class of M over $\mathbb{Q}$, consistent with its orientation. We denote it by $\sigma(M)$. If $\dim M \not\equiv 0$ (4), we define $\sigma(M) := 0$.

From now on, given a manifold M, we will write simply M to refer to its cobordism class.

Lemma 3.6. *The map* $M \longmapsto \sigma(M)$ *determines an algebra homomorphism between* $\Omega_\star^{\mathrm{SO}} \otimes \mathbb{Q}$ *and* $\mathbb{Q}$.

Lemma 3.6 is a consequence of three important properties of the signature: σ is an additive and multiplicative function with respect to the corresponding operations between cobordism classes, *i.e.* $\sigma(M_1 + M_2) = \sigma(M_1) + \sigma(M_2)$ and $\sigma(M_1 \times M_2) = \sigma(M_1)\sigma(M_2)$; and σ is a *cobordism invariant*, that is, if M_1 and M_2 belong to the same cobordism class, then $\sigma(M_1) = \sigma(M_2)$ (alternatively $\sigma(M) = 0$ if $\overline{M} = \overline{\varnothing}$).

To establish Hirzebruch's theorem it is necessary to introduce the notion of multiplicative sequences:

Definition 3.7. Let R be a commutative ring with unity and let $A^\star := \bigoplus_{i=0}^{\infty} A_i$ be a graded commutative (in the classical sense) R-algebra. Let A^Π be the set of formal series $a_0 + a_1 + a_2 + \cdots$, with each $a_i \in A_i$, and define $A_1^\Pi := \{a \in A^\Pi \mid a_0 = 1\}$. Let $\{K_n(x_1, \dots, x_n)\}_{n \geq 1}$ be a sequence of polynomials with coefficients in R s.t. $K_n(x_1, \dots, x_n)$ is homogeneous of degree n with $\dim x_i = i$ for each i. Given $a \in A_1^\Pi$, define

$K(a) := 1 + K_1(a_1) + K_2(a_1, a_2) + \cdots \in A_1^{\prod}$. We say that $\{K_n\}_{n\geq 1}$ is a *multiplicative R-sequence*, if, for any R-algebra $A^\star$ of the form described, $K(ab) = K(a)K(b)$ for all $a, b \in A_1^{\prod}$. We may write $K_i(a)$ to denote $K_i(a_1, \ldots, a_i)$, with $K_0(a) = 1$.

An important result regarding multiplicative sequences is the following lemma (see [5, p. 221]):

Lemma 3.8. *Let R be a commutative ring with unity, and let $f(t) = 1 + r_1 t + r_2 t^2 + \cdots$ be a formal power series with coefficients in R. Then, there exists a unique multiplicative R-sequence $\{K_n\}$ s.t. $K(1+t) = f(t)$. We refer to K as the* multiplicative sequence associated to $f(t)$.

Another important cobordism invariant, arising from multiplicative sequences, and with similar properties as the signature, is the K-genus:

Definition 3.9. Let M be a compact oriented $4n$-manifold, and let $\{K_n\} =: K$ be a multiplicative $\mathbb{Q}$-sequence. We define the *K-genus of M*, $K[M]$, as the number $\langle K_n(p_1, \ldots, p_n), \mu_{4n}\rangle$, where $p_j := p_j(TM)$ for each j, and μ_{4n} is the fundamental class of M. If the dimension of M is not a multiple of 4, $K[M] := 0$.

Note that the K-genus is just a rational combination of some Pontrjagin numbers of M. Since Pontrjagin numbers are additive under cobordism sum (which can be easily proven using that they are cobordism invariants and also $H^i(M_1 \sqcup M_2) \cong H^i(M_1) \oplus H^i(M_2)$), it is clear that $M \longmapsto K[M]$ is also additive. It can also be seen that $K[M_1 \times M_2] = K[M_1]K[M_2]$, using a much harder result from homological algebra known as Künneth theorem. Thus, similar to the case of the signature, we have that:

Lemma 3.10. *Given a multiplicative $\mathbb{Q}$-sequence K, the map $M \longmapsto K[M]$ determines an algebra homomorphism between $\Omega_\star^{SO} \otimes \mathbb{Q}$ and $\mathbb{Q}$.*

Theorem 3.11 (Hirzebruch's signature theorem [5, §19])**.** *Let $L := \{L_n\}$ be the multiplicative sequence associated to the power series:*

$$f(x) = \frac{\sqrt{x}}{\tanh\sqrt{x}} = 1 + \frac{1}{3}x - \frac{1}{45}x^2 + \cdots + \frac{(-1)^{n-1}2^{2n}B_n x^n}{(2n)!} + \cdots \tag{15}$$

Then, the signature, $\sigma(M)$, of a compact oriented manifold M, equals the L-genus of M, $L[M]$. Hence, for a compact oriented $4n$-manifold M, we have:

$$\sigma(M) = \langle L_n(p_1, \ldots, p_n), \mu_{4n}\rangle. \tag{16}$$

Proof. By Lemmas 3.6 and 3.10, both the signature and the L-genus define homomorphisms between $\Omega_\star^{SO} \otimes \mathbb{Q}$ and $\mathbb{Q}$. Thus, it suffices to prove the result for a set of generators of $\Omega_\star^{SO} \otimes \mathbb{Q}$. Using Theorem 3.4(ii), it is clear that we only have to show that $\sigma(\mathbb{CP}^{2n}) = L[\mathbb{CP}^{2n}]$, $n \geq 1$. Since $H^{2n}(\mathbb{CP}^{2n}; \mathbb{Q})$ is generated by t^n, with $t = -c_1(\gamma_n^1)$, it is clear that $\sigma(\mathbb{CP}^{2n}) = \langle t^{2n}, \mu_{4n}\rangle$. On the other hand, by Proposition 2.9, the total Pontrjagin class of $\mathbb{CP}^{2n}$ is $(1+t^2)^{2n+1}$. Now, since L is the multiplicative sequence associated to $f(x) = \sqrt{x}/\tanh\sqrt{x}$, by Lemma 3.8: $L((1+t^2)^{2n+1}) = L(1+t^2)^{2n+1} = (t/\tanh t)^{2n+1}$, where we use dim $t^2 = 1$ in the $\mathbb{Q}$-algebra $\bigoplus_{i=0}^{\infty} H^{4i}(\mathbb{CP}^{2n}; \mathbb{Q})$. Thus:

$$L[\mathbb{CP}^{2n}] = \langle L_n(p_1, \ldots, p_n), \mu_{4n}\rangle = \left\langle \left(\tfrac{t}{\tanh t}\right)^{2n+1}, \mu_{4n}\right\rangle = C\langle t^{2n}, \mu_{4n}\rangle = C\sigma(\mathbb{CP}^{2n}), \tag{17}$$

where C is the coefficient of t^{2n} in the power series of $(t/\tanh t)^{2n+1}$. Note that C coincides with the -1 degree coefficient of the power series around 0 of the complex function $1/(\tanh z)^{2n+1}$. Hence, by the residue theorem:

$$C = \frac{1}{2\pi i}\oint_{\gamma} \frac{1}{(\tanh z)^{2n+1}}\, dz, \tag{18}$$

with γ a sufficiently small, positively oriented, regular closed path around the origin. Consider now the substitution $u := \tanh z$, with $dz = du/(1-u^2)$, and note that:

$$C = \frac{1}{2\pi i}\oint_{\gamma'} \frac{1}{u^{2n+1}}\frac{du}{1-u^2} = \frac{1}{2\pi i}\oint_{\gamma'} \frac{1+u^2+u^4+\cdots}{u^{2n+1}}\, du = +1, \tag{19}$$

where we are again using the residue theorem. This concludes the proof, since $L[\mathbb{CP}^{2n}] = C\sigma(\mathbb{CP}^{2n})$. $\square$

4. Exotic structures

One of the major mathematical achievements of the second half of the 20th century was the discovery of topological spheres that were not diffeomorphic to the standard sphere. The first examples, the exotic 7-spheres, were unveiled by J. Milnor in 1956 (see [4]). This had profound implications in the field of manifold and algebraic topology, since, until Milnor's paper, a fundamental difference between topological and differentiable spheres was not expected. The purpose of this section is to introduce Milnor's construction and to link what has been exposed so far with the existence of non-standard smooth structures in $\mathbb{S}^7$. Milnor uses G-bundles, which are based on the concept of vector bundles, but allow fibers to be arbitrary topological spaces, connected through transition functions of the form $(b,x) \longmapsto (b, g_{ij}(b)x)$, where $g\colon U_i \cap U_j \longrightarrow G$ is a continuous map and G is a subgroup of homeomorphisms from F (the base fiber) to itself. For G-bundles over $\mathbb{S}^n$, there is a classification theorem, analogous to the one presented for vector bundles, which establishes a one-to-one correspondence between bundle isomorphism classes and homotopy classes in $\pi_{n-1}(G)$. As $\pi_3(\mathrm{SO}(4)) \cong \mathbb{Z}\oplus\mathbb{Z}$ (see [7]), one can parametrize $\mathrm{SO}(4)$-bundles over $\mathbb{S}^4$ with fiber $\mathbb{S}^3$ by two integers, and denote a set of representatives by ζ_{hj}, $h, j \in \mathbb{Z}$. One can also write for the corresponding total spaces, $E(\zeta_{hj}) =: M_{hj}$, an explicit covering set of local charts, together with their transition functions. This allows Milnor to apply a deep result from Reeb (see [6, p. 11]) to show that if $h+j=1$ and $h-j=k$, with k odd, then:

$$M_k := M_{\frac{1+k}{2},\frac{1-k}{2}} \text{ is homeomorphic to } \mathbb{S}^7.$$

Let now M be a 7-manifold, oriented by $\mu \in H_7(M)$, s.t. $H^3(M) = H^4(M) = 0$ (with coefficients in $\mathbb{Z}$ from now on). Let B be a 8-manifold s.t. ∂B and M are diffeomorphic through an orientation-preserving diffeomorphism. Then, from the long exact sequence of the pair (B, M) we deduce that $j^\star\colon H^4(B,M) \longrightarrow H^4(B)$ is an isomorphism, which allows us to define: $q(B) := \langle (j^\star)^{-1}(p_1(TB))^2, \nu\rangle$, where $\nu \in H_8(B,M)$ is the fundamental class of the pair (B,M), compatible with the orientation of M, *i.e.* $\partial\nu = \mu$. Define also $\tau(B)$ as the signature of the quadratic form over $H^4(B,M)/\text{torsion}$, given by $\alpha \longmapsto \langle \alpha^2, \nu\rangle$. Under these conditions, the following is true:

Proposition 4.1. *The residue of $2q(B) - \tau(B)$ modulo 7 is independent of B.*

This is a direct consequence of Hirzebruch's signature theorem, and is essential to prove the existence of a non-standard smooth structure for $\mathbb{S}^7$.

Sketch of proof (Proposition 4.1). Let B_1, B_2 be two 8-manifolds s.t. $\partial B_1 = \partial B_2 = M$. Define $C := B_1 \sqcup (-B_2)/\sim$, where $\sim$ identifies ∂B_1 with $\partial(-B_2)$. Since C is a closed 8-manifold, we can apply Hirzebruch's signature theorem to see that:

$$\sigma(C) = \langle \tfrac{1}{45}(7p_2(C) - p_1^2(C)), \nu \rangle, \tag{20}$$

where ν is an orientation for C compatible with the corresponding orientations, ν_1 and $-\nu_2$, for B_1 and $-B_2$ respectively. This implies $2\langle p_1(C)^2, \nu\rangle - \sigma(C) \equiv 0$ (7). It is not difficult to see, through some homological algebra computations and using the aforementioned conditions, that the quadratic form associated to C, over $H^4(C)$, is the direct sum of the quadratic forms over $H^4(B_1, M)$ and $H^4(B_2, M)$, as defined above, reversing the sign of the latter. This clearly implies $\sigma(C) = \tau(B_1) - \tau(B_2)$ and, similarly, $\langle p_1(C)^2, \nu\rangle = q(B_1) - q(B_2)$, which proves the statement. $\square$

Thus, under the stated conditions, we can define $\lambda(M) := 2q(B) - \tau(B) \in \mathbb{Z}/7$. This invariant provides a simple criterion to determine whether or not M and $\mathbb{S}^7$ can be diffeomorphic:

Proposition 4.2. *If $\lambda(M) \neq 0$, M cannot be diffeomorphic to the boundary of an 8-manifold B with $H^4(B) = 0$. In particular, if $\lambda(M) \neq 0$, M and $\mathbb{S}^7$ are not diffeomorphic.*

Since $H^4(B) \cong H^4(B, M)$, the quadratic form over $H^4(B, M)$/torsion is 0, which yields $\tau(B) = q(B) = 0$ and, consequently, $\lambda(M) = 0$. If M and $\mathbb{S}^7$ are diffeomorphic, we can choose $B = \mathbb{D}^8$ and use $H^4(\mathbb{D}^8) = 0$ to conclude. Let now M be one of the aforementioned M_{hj}. It is clear that the total space, N_{hj}, of the SO(4)-bundle, η_{hj}, that results from substituting the $\mathbb{S}^3$ fibers in ζ_{hj} by 4-disks, $\mathbb{D}^4$, satisfies $\partial N_{hj} = M_{hj}$. A rather complex computation shows that $p_1(TN_{hj}) = c(h-j)\beta$, where $c \in \mathbb{Z}$ and β is a generator of $H^4(N_{hj}) \cong \mathbb{Z}$. This can be further used, for the spaces M_k, to show that $q(N_k) = \delta c^2 k^2$, with $\delta = \pm 1$. Finally, noting that $\tau(N_k) = \delta$, it is clear that $\lambda(M_k) \equiv \delta(2c^2k^2 - 1)$ (7). Since the squares in $\mathbb{Z}/7$ are $0, 1, 2, 4$, we may choose $k = 0, 3, 1, 5$ to obtain $\lambda(M_k) \neq 0$. This proves the following:

Theorem 4.3. *$\mathbb{S}^7$ admits at least one non-standard (exotic) smooth structure.*

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