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Scheme of pairs of matrices with vanishing commutator

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Resum *(CAT)*
En aquest treball estudiarem l'esquema de parells de matrius $n \times n$ amb commutador nul, pel que es conjectura que és reduït, Cohen–Macaulay i normal. Demostrarem que és regular en codimensió 3 però no en codimensió 4. També aportem resultats similars per a altres esquemes relacionats amb el nostre esquema original. En una segona part del treball estudiem les singularitats de l'esquema de parelles de matrius que commuten a partir de l'estudi dels corresponents esquemes de jets i altres invariants de singularitats com el log-canonical threshold.

Abstract *(ENG)*
In this work we will study the scheme of $n \times n$ matrices with vanishing commutator, which is conjectured to be reduced, Cohen–Macaulay and normal. We will prove that it is regular in codimension 3 but not in codimension 4. We will also bring similar results for other schemes related to our original one. In a second part of the paper, we study the singularities of the scheme of pairs of commuting matrices from the study of the corresponding jet schemes and other singularity invariants such as the log-canonical threshold.

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1. Introduction

The aim of this work is to study the scheme of pairs of $n \times n$ matrices over an algebraically closed field K with vanishing commutator.

Definition 1.1. Let K be an algebraically closed field. For any integer $n \geq 1$, consider the scheme associated to the following set with the natural scheme structure,

$$X_n = \{(A, B) \in \mathrm{Mat}(n, K)^{\times 2} \mid [A, B] = 0\},$$

where $[A, B] = AB - BA$, and we consider $\mathrm{Mat}(n, K)^{\times 2}$ as an affine $2n^2$-dimensional space, where A and B are generic matrices. Throughout the text, we refer to this scheme as the *commuting scheme*[1] which we will also denote as X_n. Its reduced associated scheme is usually referred to as the *commuting variety* (see [7], [10], [16]) or the *variety of commuting matrices*.

Equivalently, $X_n = \operatorname{Spec} R_n/I_n$ where $R_n = K[\{a_{i,j}, b_{i,j}\}_{1\leq i,j\leq n}]$, for the matrices $A = (a_{i,j})_{1\leq i,j\leq n}$, $B = (b_{i,j})_{1\leq i,j\leq n}$, and the ideal $I_n = (f_{i,j})_{1\leq i,j\leq n}$ is generated by

$$f_{i,j} = \begin{cases} \sum\limits_{\substack{k=1\\ k\neq i}}^{n} (a_{i,k}b_{k,i} - a_{k,i}b_{i,k}) & \text{if } i = j, \\ \sum\limits_{\substack{k=1\\ k\notin\{i,j\}}}^{n} (a_{i,k}b_{k,j} - a_{k,j}b_{i,k}) + a_{i,j}(b_{j,j} - b_{i,i}) - b_{i,j}(a_{j,j} - a_{i,i}) & \text{if } i \neq j. \end{cases}$$

Remark. $\{f_{i,j}\}_{i\neq j} \cup \{f_{i,i}\}_{i\neq k}$ is a generating set of I_n for any k and has a minimal number of generators.

An important property of X_n, first proven by Motzkin and Taussky [8] (as well as a bit later by Gerstenhaber [2]), is the following theorem:

Theorem 1.2. *X_n is irreducible and of dimension $n^2 + n$ for all $n \geq 1$.*

Moreover, there is a long standing conjecture atributed to M. Artin and M. Hochster[2] (cf. [6], [11], [7], [1], [12], [13]) on the properties of X_n:

Conjecture 1.3. *X_n is reduced, Cohen–Macaulay and normal for all $n \geq 1$.*

This conjecture is actually a specific case, for $\mathfrak{g} = \mathfrak{gl}_n$, of the following one:

Conjecture 1.4. *Let $\mathfrak{g}$ be a reductive Lie algebra. Then, the associated scheme to*

$$\mathcal{C}(\mathfrak{g}) = \{(a, b) \in \mathfrak{g} \mid [a, b] = 0\}$$

is reduced, irreducible, Cohen–Macaulay and normal.

[1] We use this nomenclature as a parallelism with the use of *commuting variety* for the reduced associated scheme.

[2] It is cited as being posed by M. Artin and M. Hochster in 1982 ([6], [11], [7]), but none of the references cites those two authors directly and we have not been able to find a direct source that supports it.

Even though we know of the existence of this wider conjecture, we will only focus on the specific case of X_n.

It is interesting to remember that the properties over the scheme can be checked over the associated ring and, for that, we can use Serre's conditions:

Definition 1.5. Given a Noetherian commutative ring A and an integer $k \geq 0$, A is said to fulfil Serre's condition if

(i) R_k if $A_\mathfrak{p}$ is a regular local ring for any prime ideal $\mathfrak{p} \subset A$ such that $\operatorname{height}(\mathfrak{p}) \leq k$.

(ii) S_k if depth $A_\mathfrak{p} \geq \inf\{k, \operatorname{height}(\mathfrak{p})\}$ for any prime $\mathfrak{p}$.

Theorem 1.6 (Serre's criteria). *Given a Noetherian commutative ring A, then*

(i) A is reduced iff A satisfies R_0 and S_1;

(ii) A is normal iff A satisfies R_1 and S_2;

(iii) A is Cohen–Macaulay iff A satisfies S_k for all $k \geq 0$;

(iv) A is regular iff A satisfies R_k for all $k \geq 0$.

Other questions that can be asked are related to the singularities of these schemes. In this sense, it is thought to have rational singularities (in characteristic 0)[3], though maybe the conjecture could be about whether they have log-canonical or log-terminal singularities, and the equivalents in characteristic $p > 0$, F-rational, F-pure or strongly F-regular. These properties, in characteristic 0, can be studied through the associated jet schemes, so we will take a look at them in the last section.

On another matter, those are not easy problems, so one ends up questioning oneself about similar schemes. In our case, we studied, among others, the pairs of matrices whose commutator's diagonal vanishes, that is, the scheme associated to:

$$X_{\text{diag}} = \{(A, B) \in \operatorname{Mat}(K)^{\times 2} \mid \operatorname{diag}([A, B]) = 0\},$$

where $\operatorname{diag}(M)$ applied to a matrix M is the projection onto the diagonal elements, (i.e., $M = (m_{i,j})_{1 \leq i,j \leq n} \mapsto \operatorname{diag}(M) = (m_{i,i})_{1 \leq i \leq n}$).

2. Scheme of commuting matrices

In this section we present the results that we obtained on the *commuting scheme*. First of all, we point out that Conjecture 1.3 is known to be true for small n:

Proposition 2.1 (see [4], [5]). *X_n is reduced, irreducible and Cohen–Macaulay but not Gorenstein for $n \leq 4$.*

The proof of this result was obtained using the computational algebra system Macaulay2 ([3]). In that matter, we have redone the computations with a small improvement that might be helpful in attempting the proof for $n = 5$.

[3]The statement of rational singularities is not a published conjecture or open problem, but it would fit in the behaviour of a more general family of schemes that are closely related to it, studied in [1].

Proposition 2.2. *$\mathcal{O}_{X_n} := R_n/I_n$ is Cohen–Macaulay (respectively reduced) iff, for any $1 \leq i, j \leq n$, the quotient $\mathcal{O}_{X_n}/(a_{i,i}, b_{j,j})$ is Cohen–Macaulay (respectively reduced). Where $(a_{i,i}, b_{j,j})$ is the ideal (sheaf) generated by the (i,i)-th entry of the matrix A and the (j,j)-th entry of the matrix B.*

Furthermore, we have proven the following result:

Theorem 2.3. *X_n is regular in codimension 3 but not 4 for all $n \geq 1$. That is, it satisfies Serre's conditions R_0, R_1, R_2 and R_3 but not R_k for any $k \geq 4$.*

This result has the following implications:

Proposition 2.4. *The singular locus of X_n^{red}, the associated reduced scheme of X_n, has codimension at least 4. If X_n is reduced, then its singular locus has codimension 4.*

Proposition 2.5. *If X_n has any embedded component, it must have at most dimension $n^2 + n - 4$.*

In particular, Theorem 2.3 implies, through Serre's criteria (Theorem 1.6), the following proposition:

Proposition 2.6. *If X_n is Cohen–Macaulay, then it is reduced and normal.*

The implication of being reduced was known previously (cf. [4]), but the argumentation was different (see [15, Prob. 2.7.1]). The implication of being normal was also known as an implication of it being reduced and the following theorem:

Theorem 2.7 ([12])**.** *Given a connected non-commutative reductive lie algebra $\mathfrak{g}$ over an algebraically closed field K of characteristic 0, let $\mathcal{C}^{\text{red}}(\mathfrak{g}) = \{(a,b) \in \mathfrak{g} \mid [a,b] = 0\}$ be the reduced scheme of pairs of commuting elements. Then $\operatorname{codim}_{\mathfrak{g}\times\mathfrak{g}}(\mathcal{C}^{\text{red}}(\mathfrak{g}))^{\text{sing}} \geq 2$, where $(\mathcal{C}^{\text{red}}(\mathfrak{g}))^{\text{sing}}$ stands for the singular locus of $\mathcal{C}^{\text{red}}(\mathfrak{g})$.*

Even though Proposition 2.6 can be deduced from results that were already known, its implications to X_n for $n \leq 4$ do not seem to be recorded in the literature. In any case, we have:

Proposition 2.8. *X_n is reduced, irreducible, Cohen–Macaulay and normal, but not Gorenstein, for $n \leq 4$.*

The proof of Theorem 2.3 is too long to be included in its full extension, so we will just give the main ideas.

Sketch of Proof of Theorem 2.3. For ease of reading we have divided the proof in three parts. Throughout we will use the Jacobian smoothness criterion.

1. *R_0 and R_1 properties.*

   Let us consider B in Jordan canonical form. If we name J_k the nilpotent Jordan block of size k, then there exist $\lambda_1, \dots, \lambda_r \in K$ pairwise different elements and $a_1, \dots, a_r > 0$ integers satisfying $a_1 + \cdots + a_r = n$, such that B is a block diagonal matrix of the form $B = \operatorname{diag}(\lambda_1 I_{a_1} + J_{a_1}, \dots, \lambda_r I_{a_r} + J_{a_r}) = (b_{i,j})_{1 \leq i,j \leq n}$.

   In this case:

$$c_{i,j}^{r,s} := \frac{\partial f_{r,s}}{\partial a_{i,j}} = \begin{cases} 1 & \text{if } i = r,\ s = j+1 \leq n \text{ and } b_{j,j} = b_{j+1,j+1}, \\ -1 & \text{if } j = s,\ r = i-1 \geq 0 \text{ and } b_{i-1,i-1} = b_{i,i}, \\ b_{j,j} - b_{i,i} & \text{if } (i,j) = (r,s) \text{ and } b_{j,j} \neq b_{i,i}, \\ 0 & \text{otherwise.} \end{cases}$$

First, we will prove that $\det(c^{r,s}_{i,j})_{\substack{b_{r,r}\neq b_{s,s}\\ b_{i,i}\neq b_{j,j}}} \notin I_n$, where the columns of the matrix are indexed by the (i,j) and the rows by (r,s), both with the same ordering. We observe that the product of the diagonal elements is $\prod_{\{(i,j)|b_{i,i}\neq b_{j,j}\}} (b_{j,j} - b_{i,i}) \notin I_n$. We will prove that all the other products in the determinant vanish.

Let us pick the column (i,j) and assume that we have to pick a nonzero element outside the diagonal. If $j+1 \leq n$ and $b_{j,j} = b_{j+1,j+1}$, then $b_{i,i} \neq b_{j+1,j+1}$, so for the (i,j) column, we can get the entry of the $(i,j+1)$ row which has a value of 1. In this case, for the $(i,j+1)$ column we cannot get the diagonal element. If $i-1 \geq 0$ and $b_{i-1,i-1} = b_{i,i}$, then $b_{i-1,i-1} \neq b_{j,j}$ and for the (i,j) column we can get the entry of the $(i-1,j)$ row that has a value of -1. In this case, for the $(i-1,j)$ column we cannot get the diagonal element. Otherwise, the only nonzero element is the diagonal one.

A non-vanishing product would be equivalent to this process having a cycle, but either the i decreases or the j increases, so we can never have a cycle, and all products, apart from the diagonal one, vanish, as we wanted to show.

Now, we will reason by induction. Given (k,l) such that $b_{k,k} = b_{l,l}$, $l+1 \leq n$ and $b_{l,l} = b_{l+1,l+1}$, assume that all the columns with indexes in

$$\mathcal{S} = \{(i,j) \mid b_{i,i} \neq b_{j,j}\} \cup \{(i,j) \mid b_{i,i} = b_{j,j},\ j+1 \leq n,\ b_{j,j} = b_{j+1,j+1} \text{ and } (i,j) < (k,l)\},$$

where the ordering is the lexicographic order, are linearly independent. Then, $c^{k,l+1}_{k,l} = 1$ and for all $(i,j) \in \mathcal{S}$, $c^{k,l+1}_{i,j} = 0$, which proves that the columns with indexes in $\mathcal{S} \cup \{(k,l)\}$ are linearly independent. In this way, we have proven that the columns with indexes in

$$\mathcal{I} = \{(i,j) \mid b_{i,i} \neq b_{j,j}\} \cup \{(i,j) \mid b_{i,i} = b_{j,j},\ j+1 \leq n,\ b_{j,j} = b_{j+1,j+1}\}$$

are linearly independent.

Since the cardinality of $\mathcal{I}$ is $n^2 - n$, we get that this closed point is reduced.

Through the action of $\mathrm{GL}_n(K)$ we get that the open set that includes all closed points (A,B) where B is non-derogatory is regular.

Since the complementary of the set where A and B are non-derogatory can be checked to have codimension 2, this implies R_0 and R_1 for X_n.

2. *R_2 and R_3 properties.*

   First of all, we notice:
$$X_n^{\mathrm{red}} = Y \cup \bigcup Y^{r,s}_{\substack{(i_{1,1},\dots,i_{1,t_1},i_{2,1},\dots,i_{2,t_2},\dots,i_{r,t_r})\\(j_{1,1},\dots j_{1,t'_1},j_{2,1},\dots j_{2,t'_2},\dots j_{r,t'_r})}},$$

   where $Y = \{(A,B) \in X_n^{\mathrm{red}} \mid A \text{ and } B \text{ are non-derogatory}\}$ and

$$Y^{r,s}_{\substack{(i_{1,1},\dots,i_{1,t_1},i_{2,1},\dots,i_{2,t_2},\dots,i_{r,t_r})\\(j_{1,1},\dots j_{1,t'_1},j_{2,1},\dots j_{2,t'_2},\dots j_{r,t'_r})}}$$

   is the set of pairs of commuting matrices (A,B) such that both are derogatory, A has r distinct generalised eigenvalues with Jordan decomposition in blocks of sizes $(i_{1,1},\dots,i_{1,t_1},i_{2,1},\dots,$

$i_{2,t_2}, \dots, i_{r,t_r})$, and B has s different generalised eigenvalues with Jordan decomposition in blocks of sizes $(j_{1,1}, \dots, j_{1,t'_1}, j_{2,1}, \dots, j_{2,t'_2}, \dots, j_{r,t'_r})$.

If $r \leq n-3$ or $s \leq n-3$, then

$$\dim\left(Y^{r,s}_{\substack{(i_{1,1},\dots,i_{1,t_1},i_{2,1},\dots,i_{2,t_2},\dots,i_{r,t_r})\\(j_{1,1},\dots,j_{1,t_1},j_{2,1},\dots,j_{2,t_2},\dots,j_{r,t_r})}}\right) \leq n-4,$$

for any $(i_{1,1}, \dots, i_{1,t_1}, i_{2,1}, \dots, i_{2,t_2}, \dots, i_{r,t_r})$ and any $(j_{1,1}, \dots, j_{1,t'_1}, j_{2,1}, \dots, j_{2,t'_2}, \dots, j_{r,t'_r})$, so we can ignore those sets.

Then, to prove the result, it is enough to check, for each one of the sets corresponding to $n-2 \leq r, s \leq n-1$, either that it has, at most, dimension $n-4$, or that it is composed of regular points. Checking for regularity is done by computing the rank of the Jacobian matrix.

3. *R_4 property failure.*

   Take the closed points of the form (A, B) where A and B are both diagonalisable and they both have $n-1$ distinct eigenvalues, such that, when simultaneously diagonalised, they have the form $gAg^{-1} = \mathrm{diag}(\lambda_2, \lambda_2, \lambda_3, \lambda_4, \dots, \lambda_n)$, $gBg^{-1} = \mathrm{diag}(\mu_2, \mu_2, \mu_3, \mu_4, \dots, \mu_n)$, for certain $g \in \mathrm{GL}_n(K)$ and certain $\lambda_i, \mu_j \in K$. It is immediate to check that the Jacobian matrix has rank at most $n^2 - n - 2$, so these are all non-regular points. On the other hand, the codimension is 4.

□

2.1 Related schemes

As we stated in the introduction, we have also worked with some similar schemes, which has lead to the solution of a small open problem posed by Hsu-Wen Young in his PhD dissertation [16]:

Theorem 2.9. *Given a field K, the scheme associated to $X = \{(A, B) \in \mathrm{Mat}(n, K)^{\times 2} \mid \mathrm{diag}([A, B]) = 0\}$, where $\mathrm{diag}(M)$ applied to a matrix M is the projection onto the diagonal elements (i.e., $M = (m_{i,j})_{1 \leq i,j \leq n} \mapsto \mathrm{diag}(M) = (m_{i,i})_{1 \leq i \leq n}$), is a reduced irreducible normal complete intersection scheme over K.*

Hsu-Wen Young proved it to be a reduced complete intersection for general n and checked it to be irreducible for $n \leq 3$. His motivation was mainly as a counterpart to the *diagonal commutator scheme*, which is the scheme:

$$D_n = \{(A, B) \in \mathrm{Mat}(n, F)^{\times 2} \mid [A, B] = \mathrm{diag}([A, B])\},$$

that is, the pairs of matrices whose commutator is diagonal.

The proof of Theorem 2.9 follows from an easy induction, the Jacobian smoothness criterion and the use of the following lemmas:

Lemma 2.10. *If R is a ring, and $a \in R$ is not a zero-divisor, then R is a domain (respectively reduced) if and only if R_a is a domain (respectively reduced).*

Remark. This implies that if we have an element $a \in R$ and an ideal such that $(I : (a)) = I$, I is prime (resp. radical) iff it is prime (resp. radical) in R_a (thanks to the localisation at a multiplicative set S being an exact functor from R-modules to $S^{-1}R$-modules).

Lemma 2.11. *Given a ring R, it is a domain (respectively reduced) iff the polynomial ring $R[X]$ is a domain (respectively reduced).*

3. Jet schemes

In this section we will study the jet schemes over X_n, which are known to be closely related to its singularities and which will allow us to get some results on the log-canonical threshold, another singularity invariant.

Definition 3.1. The *m-th jet scheme* associated to a scheme X over an algebraically closed field K is the set $X^{(m)}(K) = \mathrm{Hom}_K(\mathrm{Spec}(K[t]/t^{m+1}), X)$ with a natural scheme structure.

It is a well known result that the jet schemes over an affine scheme are again affine. Furthermore, there is the following result:

Theorem 3.2. *Given a field K and an affine scheme $X = \mathrm{Spec}(K[x_1, \dots, x_n]/I)$ over K, where $I = (f_1, \dots, f_r) \subset K[x_1, \dots, x_n]$ is an ideal, we have that the defining equations for the m-th jet scheme over the polynomial ring $K[\{x_{1,k}, \dots, x_{n,k}\}_{0\leq k\leq m}]$ are*

$$\begin{gathered} f_1(\tilde{x}_1(t), \dots, \tilde{x}_n(t)) \cong 0 \mod t^{m+1}, \\ \vdots \\ f_r(\tilde{x}_1(t), \dots, \tilde{x}_n(t)) \cong 0 \mod t^{m+1}, \end{gathered}$$

where $\tilde{x}_i(t) = x_{i,0} + x_{i,1}t + \cdots + x_{i,m}t^m$.

Applied to our scheme, we get:

Proposition 3.3. *Over the ring $K\left[\{a_{i,j,k}, b_{i,j,k}\}_{\substack{0\leq k\leq m\\ 1\leq i,j\leq n}}\right]$, we define the matrices $A_k = (a_{i,j,k})_{1\leq i,j\leq n}$, $B_k = (b_{i,j,k})_{1\leq i,j\leq n}$. In this situation, the elements generating the ideal that defines the m-th jet scheme over X_n, which we name $X_n^{(m)}$, are the entries of the following matrices:*

$$\begin{gathered} [A_0, B_0] \\ [A_0, B_1] + [A_1, B_0] \\ [A_0, B_2] + [A_1, B_1] + [A_2, B_0] \\ \dots \\ [A_0, B_m] + [A_1, B_{m-1}] + \cdots + [A_{m-1}, B_1] + [A_m, B_0]. \end{gathered}$$

Remark. It is worth noticing that the group $\mathrm{GL}_n(K)$ acts on the scheme by simultaneous conjugation on all the matrices $X_0, \dots, X_m$, $Y_0, \dots, Y_m$.

The main results known about the jet schemes of X_n are:

Theorem 3.4 ([14])**.** *For $n \leq 3$ and for all $m \geq 0$, the m-th jet scheme over X_n is irreducible and of dimension $(n^2 + n)(m + 1)$.*

Theorem 3.5 ([14])**.** *For all $m > 0$ exists an integer $N(m)$ such that for all $n \geq N(m)$, the m-th jet scheme over X_n is reducible.*

Even though it is not mentioned in that paper, the following proposition can be deduced from the proof of Theorem 3.5:

Proposition 3.6. *For all $m > 0$ there exists an integer $N(m)$ such that for all $n \geq N(m)$, the m-th jet scheme over X_n is not equidimensional and of dimension strictly greater than $(n^2 + n)(m + 1)$.*

Now, it is worth noticing the following result by Mustaţă:

Theorem 3.7 ([9])**.** *If X is a smooth variety over $\mathbb{C}$ and $Y \subset X$ is a closed sub-scheme, then the log-canonical threshold of the pair (X, Y) is given by*

$$\operatorname{lct}(X, Y) = \dim X - \sup_{m \geq 0} \frac{\dim Y^{(m)}}{m + 1},$$

where $Y^{(m)}$ represents the m-th jet scheme over Y.

Joining these all, we obtain the following:

Proposition 3.8. *For $n \leq 3$, $\operatorname{lct}(\operatorname{Mat}(n, \mathbb{C})^{\times 2}, X_n) = n^2 - n = \operatorname{codim} X_n$.*

Proposition 3.9. *For $n \geq 30$, $\operatorname{lct}(\operatorname{Mat}(n, \mathbb{C})^{\times 2}, X_n) < n^2 - n = \operatorname{codim} X_n$.*

These results show the differences in the behaviour of the singularities of X_n depending on n.

The main results of Sethuraman and Šivic come from the existence of an irreducible open set of $X_N^{(m)}$ having dimension $(n^2+n)(m+1)$, which we denote $U_n^{(m)}$, formed by the set of closed points $(A(t), B(t)) = (A_0 + A_1 t + \cdots + A_m t^m, B_0 + B_1 t + \cdots + B_m t^m)$ where A_0 is non-derogatory, and the following lemmata:

Lemma 3.10. *Given a positive integer N, assume that $X_n^{(m)}$ is irreducible for all $n < N$. Then, for any point $(A, B) = (A(t), B(t)) \in X_N^{(m)}$ such that A_0 or B_0 have two distinct eigenvalues, we have that $(A, B) \in \overline{U}_N^{(m)}$, where $\overline{U}_N^{(m)}$ denotes the closure of $U_N^{(m)}$.*

And, if we define the corresponding open set where B_0 is non-derogatory as $U_n'^{(m)}$:

Lemma 3.11. *Let f be an automorphism of $X_n^{(m)}$ such that $f(U_n^{(m)}) = U_n^{(m)}$ or $f(U_n'^{(m)}) = U_n'^{(m)}$ or $f(U_n^{(m)} \cap U_n'^{(m)}) = U_n^{(m)} \cap U_n'^{(m)}$. Then, $(A, B) \in \overline{U}_n^{(m)}$ iff $f(A, B) \in \overline{U}_n^{(m)}$.*

Our method consists in proving that the closed subvariety where A_0 is in a specific nilpotent Jordan canonical form is irreducible. In this case, the set

$$S_{A_0} = \{(A'(t), B'(t)) \in X_n^{(m)} \mid \exists g \in \operatorname{GL}_n(F),\, \lambda \in F \text{ such that } A_0' = gA_0g^{-1} + \lambda I\}$$

is irreducible. Finally, we have that there is a non-derogatory matrix B_0 commuting with A_0. Taking $A(t) = A_0 + 0t + \cdots + 0t^m$ and $B(t) = B_0 + 0t + \cdots + 0t^m$, we have that this pair belongs to $U_n^{(m)}$ and, therefore, $S_{A_0} \cap U_n^{(m)} \neq \varnothing$. Which, by the irreducibility of S_{A_0}, implies $S_{A_0} \subset \overline{U}_n^{(m)}$.

We also used similar methods to set bounds on the dimension of the jet schemes.

Let us define

$$Y_{(r_1,\dots,r_s)} = \{(A(t), B(t)) \in X_n^{(1)} \mid A_0 = J_{(r_1,\dots,r_s)}\},$$
$$\widetilde{Y}_{(r_1,\dots,r_s)} = \{(A(t), B(t)) \in X_n^{(1)} \mid \exists g \in \mathrm{GL}_n(F),\ \exists \lambda \in K \text{ s.t. } gA_0g^{-1} + \lambda I = J_{(r_1,\dots,r_s)}\},$$

where $J_{(r_1,\dots,r_s)}$ refers to the nilpotent matrix in Jordan canonical form with s blocks of sizes $r_1, \dots, r_s$.

The results that we obtained using the described method and basic linear algebra are the following:

Proposition 3.12. *The reduced scheme associated to*

(i) $Y_{(1,\dots,1)}$ *is irreducible for all* $n \geq 1$;

(ii) $Y_{(n/r,\dots,n/r)}$, *for* $r|n$, *is irreducible if and only if* $X_{n/r}^{(r-1)}$ *is irreducible;*

(iii) $Y_{(n-r,1,\overset{r}{\dots},1)}$, *for* $r \geq 0$ *is irreducible for all* $n \geq r+2$ *if and only if it is for some* $n \geq r+2$;

(iv) $Y_{(n-2,1,1)}$ *is irreducible for all* $n \geq 4$;

(v) $Y_{(n-r,1,\overset{r}{\dots},1)}$, *for* $r \geq 0$, *has the same codimension for all* $n \geq r+2$;

(vi) $\widetilde{Y}_{(n-r,1,\overset{r}{\dots},1)}$, *for* $r \geq 0$, *has dimension at most* $2(n^2+n)$;

(vii) $\widetilde{Y}_{((n-1)/2,(n-1)/2,1)}$, *for* $n = 5$, *has dimension at most* $2(n^2+n)$.

All these results allowed us to prove the following:

Theorem 3.13. *The first jet scheme over* X_4 *is irreducible of dimension* $2(4^2+4) = (m+1)(n^2+n)$.

Theorem 3.14. *The first jet scheme over* X_5 *has dimension* $2(5^2+5) = (m+1)(n^2+n)$.

These results on the jet schemes have implications on another open problem (see [14]) that deals with the dimension of $K[A_1, \dots, A_m]$, the algebra generated by m square $n \times n$ commuting matrices over a field K. The question is whether it is bounded by n. The answer is positive for $m = 2$ and negative for $m \geq 4$ (cf. [14]).

Specifically, Sethuraman and Šivic introduced a relation between the jet schemes over X_n with algebras generated by three commuting matrices:

Proposition 3.15 ([14])**.** *Given* K *an algebraically closed field and* $k \geq 0$ *an integer, if* J_{k+1} *is the nilpotent Jordan block of dimension* $k+1$, C *is a block diagonal matrix in* $\mathrm{Mat}(n(k+1), K)$ *consisting of* n *copies of* J_{k+1} *along the diagonal up to addition of scalars and* A, B *two matrices commuting with* C, *then if* $X_n^{(k)}$ *is irreducible* $\dim K[A, B, C] \leq n(k+1)$.

In particular, if we combine this proposition with the results that we obtained on the first jet scheme over X_4, we obtain the following new result:

Corollary 3.16. *Let* K *be an algebraically closed field. If* J_2 *is the nilpotent Jordan block of dimension* 2, C *is a block diagonal matrix in* $\mathrm{Mat}(8, K)$ *consisting of* 4 *copies of* J_2 *along the diagonal up to addition of scalars and* A, B *two matrices commuting with* C, *then* $\dim K[A, B, C] \leq 8$.

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