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K-theory for C*-algebras: The hexagonal exact sequence

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Aquest treball té com a objectiu introduir el lector a la teoria K per C^*-àlgebres demostrant-ne dos dels seus resultats centrals coneguts: La periodicitat de Bott i la successió exacta cíclica de sis termes. Aquests dos resultats constitueixen una eina essencial de cara al càlcul explícit dels K-grups d'una C^*-àlgebra, i han estat utilitzats amb èxit en l'estudi de diverses famílies. De cara a enunciar-los, ens desviem lleugerament de la literatura estàndard i introduïm la notació K', que permet simplificar els resultats i definicions necessàries per entendre les seves demostracions.

Abstract *(ENG)*

The aim of this work is to introduce the reader to C^*-algebraic K-theory whilst proving two of its main known results: Bott periodicity and the hexagonal exact sequence. These constitute a determinant tool for the explicit computation of the K-groups of a C^*-algebra, and have been used succesfully to study a variety of families. In order to state them, we deviate slightly from the standard literature and introduce the notation K', which allows us to simplify the results and definitions needed to understand their proofs.

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1. Introduction

The development of C^*-algebraic K-theory was initiated in the early 1970s, when G.A. Elliott classified the so-called approximately finite dimensional C^*-algebras by using their ordered K_0 groups, see [2]. Since then, C^*-algebraic K-theory has become an important tool in the treatment of operator algebras, and has been used succesfully to classify a considerably large family of separable and simple C^*-algebras.

In analogy to the topological K-theory developed by Aityah–Hirzebruch, in C^*-algebraic K-theory one defines a family of functors K_n from the category of C^*-algebras to that of abelian groups, thus assigning to every C^*-algebra A a family of groups $K_n(A)$. The computation of these groups, usually known as the K-groups of the algebra, provides useful information on the structure of the sets of projections and unitaries of A.

Towards this computation, and in contrast to algebraic K-theory, there exist a number of tools that make the treatment of the K-groups of a C^*-algebra manageable. Amongst them, there are two that are of particular importance: The first one, known as Bott periodicity, is the C^*-algebraic equivalent to the periodicity obtained in topological K-theory, and states that all K-groups of even and odd subscripts are isomorphic to K_0 and K_1, respectively; see [1, Ch. 9].

The second result, which is a consequence of the first one, allows us to construct a hexagonal exact sequence from any exact sequence of C^*-algebras. In particular, the existence of such a sequence implies that one can compute the K-groups of a C^*-algebra by studying the K-groups of one of its ideals and its corresponding quotient; see [1, § 9.3].

Therefore, the aim of this work is to introduce the reader to C^*-algebraic K-theory whilst proving these two results. More explicitly, for any exact sequence of C^*-algebras

$$0 \longrightarrow I \xrightarrow{\varphi} A \xrightarrow{\phi} B \longrightarrow 0, \tag{1}$$

we wish to obtain the associated hexagonal exact sequence

$$\begin{tikzcd} K_1(I) \arrow[r, "K_1(\varphi)"] & K_1(A) \arrow[r, "K_1(\phi)"] & K_1(B) \arrow[d, "\delta_1"] \\ K_0(B) \arrow[u, "\delta_0"] & K_0(A) \arrow[l, "K_0(\phi)"'] & K_0(I). \arrow[l, "K_0(\varphi)"'] \end{tikzcd} \tag{2}$$

To this end, we will assume without loss of generality (see, for example, [4, § 1.1.5]) that (1) is of the form

$$0 \longrightarrow I \xrightarrow{i} A \xrightarrow{\pi} A/I \longrightarrow 0, \tag{3}$$

where I is an ideal of A and i, π are the usual inclusion and quotient mappings.

The remainder of this paper has been divided into three parts, where some familiarity with C^*-algebras is assumed. All the C^*-algebraic background needed for these sections can be found in many textbooks; see, for example, [3].

In Section 2 we recall the definitions of the K-functors as well as some of their properties. We also introduce the notation K'_n, which will allow us to shorten the definitions of this section and the proofs of Section 3. As the aim of Section 2 is for the reader to get acquainted with the basics of K-theory, we omit all the proofs.

The index map, denoted by δ_1, is then defined in Section 3, where we first prove that the K'-functors are exact and invariant under homotopy. As these functors are equivalent to the K-functors, the construction of the index map together with this fact gives us a six-term exact sequence which is not yet cyclic.

Finally, in Section 4 we prove Bott periodicity and construct the hexagonal exact sequence by closing the sequence obtained in Section 3. Since the proofs of some of the preliminary lemmas in this section are rather long and arduous, we only provide a reference for them.

2. An overview of C*-algebraic K-theory

In this first section we briefly review the concepts and results of C^*-algebraic K-theory that will be used in the subsequent sections. All the proofs can be found in [4, 5]. Throughout this paper, A and B will denote C^*-algebras and, given any two square matrices a, b over A, we will refer to the matrix $\mathrm{diag(a, b)}$ by $a \oplus b$.

2.1 The projection group K$_0$ and the unitary group K$_1$

We begin our overview defining the first two K-groups. As we will later prove, these are the only ones up to isomorphism.

Proposition 2.1. *Let $P_n(A)$ be the sets of projections in $M_n(A)$ and denote by $P_\infty(A)$ their disjoint union. Then, by writting $p \sim_0 q$ if and only if $p = vv^*$ and $q = v^*v$ for some rectangular matrix v, one gets that $(P_\infty(A)/\sim_0, \oplus)$ is a commutative monoid with the class of 0 as its unit.*

Definition 2.2. For any unital C^*-algebra A, the group $K_0(A)$ is defined to be the Grothendieck group of the monoid above, where we denote the class of an element $p \in P_\infty(A)$ by $[p]_0$. If A does not have a unit, we define the group $K_0(A)$ as the kernel of the map $K_0(\pi)\colon K_0(\tilde{A}) \to K_0(\mathbb{C})$, where $K_0(\pi)([p]_0) = [\pi(p)]_0$ and π is the usual projection map from $\tilde{A}$ to $\mathbb{C}$ applied entry-wise (here, we use $\tilde{A}$ to denote the unitification of A).

Remark 2.3. It can be shown that every element in $K_0(A)$ is of the form $[p]_0 - [1_n \oplus 0_n]_0$ for some projection $p \in P_\infty(\tilde{A})$ whose scalar part $s(p)$ is $1_n \oplus 0_n$. Moreover, if A is unital, it follows from its construction that every element in $K_0(A)$ is of the form $[p]_0 - [q]_0$ for some $p, q \in P_\infty(A)$, where we can assume that both projections are of the same size.

As A is equipped with a norm, we can study its induced topology. In particular, we say that two elements a and b are homotopic in a subset $S \subset A$, in symbols $a \sim_h b$, if there exists a continuous path in S going from a to b. For example, given two projections p, q homotopic in $P_n(A)$, one can see that $[p]_0 = [q]_0$. Conversely, if $[p]_0 = [q]_0$, then $p \oplus 0_s \sim_h q \oplus 0_t$ for some positive integers s and t.

We will also say that two $*$-homomorphisms φ_0 and φ_1 from A to B are homotopic if there is a continuous map $t \mapsto \varphi_t$ from $[0, 1]$ to the $*$-homomorphisms from A to B such that $t \mapsto \varphi_t(a)$ is a homotopy for each $a \in A$. Moreover, two C^*-algebras A and B are said to be homotopic if there exist two $*$-homomorphisms ϕ and φ such that $\phi \circ \varphi \sim_h \mathrm{id_B}$ and $\varphi \circ \phi \sim_h \mathrm{id_A}$.

Proposition 2.4. *Let A be a unital C^*-algebra and consider the set $U_\infty(A) = \cup_n U_n(A)$, where $U_n(A)$ denotes the set of all unitary $n \times n$ matrices over A. Then, the equivalence relation "$u \sim_1 v$ if and only if $u \oplus 1_n \sim_h v \oplus 1_m$ in $U_N(A)$ for some suitable integers n, m, and N" makes $(U_\infty(A)/\sim_1, \oplus)$ into a commutative group with the class of 1 as its unit.*

Definition 2.5. Given a unital C^*-algebra A, the group $K_1(A)$ is the commutative group defined above, where we refer to the class of a unitary $u \in U_\infty(A)$ as $[u]_1$. If A does not have a unit, we define $K_1(A) := K_1(\tilde{A})$.

Remark 2.6. It can be proven that every element in $K_1(A)$ is of the form $[u]_1$ with $u \in U_\infty^+(\tilde{A})$, where $U_\infty^+(\tilde{A})$ is the set of unitaries whose scalar part is of norm 1.

Example 2.7. It is easy to see that two elements $p, q \in P_\infty(\mathbb{C})$ are equivalent under $\sim_0$ if and only if $\dim(\mathrm{Im(p)}) = \dim(\mathrm{Im(q)})$. Thus, it follows that $K_0(\mathbb{C}) \cong \mathbb{Z}$. Moreover, recall that a unitary u in a unital C^*-algebra is homotopic to 1 in $U(A)$ if and only if its spectrum is not $\mathbb{T}$; see [4, Lem. 2.1.3(ii)]. Therefore, as all unitaries in $U_\infty(\mathbb{C})$ have finite spectrum, they must be equivalent to 1 under $\sim_1$. This implies that $K_1(\mathbb{C}) = 0$. One can also adapt these arguments to see that $K_0(B(H)) = K_1(B(H)) = 0$ for any separable infinite dimensional Hilbert space H.

2.2 Suspension functor and higher index K-groups

Once the K_0 and K_1 groups have been defined, one can make use of the suspension functor S to define two families of groups: the higher index K-groups and the K'-groups. Even though these two families turn out to be the same, the introduction of the K'-groups allows us to simplify both the definitions and proofs regarding the properties of the higher index K-groups.

Recall that the suspension functor S is an exact covariant functor mapping a C^*-algebra A to $SA := \{f \in C(\mathbb{T}, A) \mid f(1) = 0\}$, and a $*$-homomorphism $\phi\colon A \to B$ to the $*$-homomorphism $S\phi$ from SA to SB defined as $S\phi(f) = \phi \circ f$.

Definition 2.8. By using the notation $S^0 = \mathrm{id}$ and $S^n = S^{n-1} \circ S$, we define the higher index K-groups $K_n(A) = K_1(S^{n-1}A)$ and the K'-groups $K_n'(A) = K_0(S^n(A))$.

Now let $\phi\colon A \to B$ be a $*$-homomorphism. We denote by $K_n'(\phi)\colon K_n'(A) \to K_n'(B)$ the group homomorphism $K_n'(\phi)([p]_0 - [s(p)]_0) = [S^n\tilde{\phi}(p)]_0 - [S^n\tilde{\phi}(s(p))]_0$. One can check that this definition makes K_n' into functors. A proof of the theorem below can be found in [5, Thm. 7.2.5].

Theorem 2.9. *Given any C^*-algebra A, consider the map $\theta_{A,n}\colon K_n(A) \to K_n'(A)$ defined as*

$$\theta_{A,n}([u]_1) = [w(1_m \oplus 0_m)w^*]_0 - [1_m \oplus 0_m]_0\,, \qquad u \in U_m^+(\widetilde{S^{n-1}A}),$$

where w is a homotopy between 1_{2m} and $u \oplus u^*$ in $U_{2m}(S^{n-1}A)$. Then, $\theta_{A,n}$ is an isomorphism for every integer $n \geq 1$.

Definition 2.10. Given a $*$-homomorphism $\phi\colon A \to B$, we define $K_n(\phi)\colon K_n(A) \to K_n(B)$ as $K_n(\phi) = \theta_{B,n}^{-1} \circ K_n'(\phi) \circ \theta_{A,n}$. Together with this definition, the K-groups also become functors.

3. Homotopy invariance and the index map

The goal of this section is to define the map δ_1 from (2) and prove that the two rows together with the right column of (2) form an exact sequence. However, we will first show that the K_n' functors are invariant under homotopy, as this is one of the main tools used in the explicit computation of the K groups of a C^*-algebra. Note that, by their definition and Theorem 2.9, this will imply that the functors K_n are also invariant under homotopy.

Theorem 3.1. *Given two homotopic $*$-homomorphisms φ_0 and φ_1 from A to B, we have that $K_n'(\varphi_0) = K_n'(\varphi_1)$, for every $n \geq 0$.*

Proof. Fix $n \in \mathbb{N}$ and let q be an element in $P_k(\widetilde{S^nA})$ for some k. Then, write q as the sum $q = p + \alpha 1_{\widetilde{S^nA},k}$ with $p \in M_k(S^nA)$ and $\alpha \in M_k(\mathbb{C})$. For every $t \in [0,1]$, define the elements $p_t = S^n\varphi_t(p)$ and $q_t = \widetilde{S^n\varphi_t}(q)$, where $t \mapsto \varphi_t$ is the homotopy from φ_0 to φ_1.

As $\widetilde{S^n\varphi_t}$ is a $*$-homomorphism, it follows that q_t is a projection for every t. Moreover, one gets that $q_t = p_t + \alpha 1_{\widetilde{S^nB},k}$ and, consequently, that $t \mapsto q_t$ is continuous if and only if $t \mapsto p_t$ is continuous.

Now let $\delta_p \colon [0,1] \times \mathbb{T}^n \to A$ be the map defined as $\delta(p)(t,(z_1,\ldots,z_n)) = p_t(z_1)(z_2)\cdots(z_n)$, and note that, for any two pairs $(t_1,\xi_1),(t_2,\xi_2) \in [0,1] \times \mathbb{T}^n$, one gets

$$\|\delta_p(t_1,\xi_1) - \delta_p(t_2,\xi_2)\| \leq \|\varphi_{t_1}(p(\xi_1)) - \varphi_{t_2}(p(\xi_1))\| + \|p(\xi_1) - p(\xi_2)\|.$$

Thus, since p is continuous and $t \mapsto \varphi_t$ is a homotopy, we have that δ_p is also continuous. Furthermore, as δ_p has compact support, the map is uniformly continuous.

It then follows that $t \mapsto p_t$ is continuous and that $t \mapsto q_t$ is a homotopy of projections. Therefore, one gets $[q_0]_0 = [q_1]_0$ for any $q \in P_\infty(\widetilde{S^nA})$, which implies the equality $K_0(\widetilde{S^n\varphi_0}) = K_0(\widetilde{S^n\varphi_1})$, from which the desired result follows. □

Example 3.2. Let X be a compact, Hausdorff, and contractible topological space. Then, the K_0 and K_1 groups of the C^*-algebra $C(X,\mathbb{C})$ are isomorphic to $\mathbb{Z}$ and 0, respectively, as $C(X,\mathbb{C})$ is homotopic to $\mathbb{C}$. Recall that X is contractible if there exists a point x_0 and a continuous map $c \colon X \times [0,1] \to X$ such that $c(x,0) = x$ and $c(x,1) = x_0$ for every $x \in X$. Then, a pair of functions giving the homotopy between $C(X,\mathbb{C})$ and $\mathbb{C}$ are $z \mapsto z1_{C(X,\mathbb{C})}$ and $f \mapsto f(x_0)$. For more details, see [4, Ex. 3.3.6].

Proposition 3.3. *For any exact sequence of the form (3), the induced sequence*

$$K_n(I) \xrightarrow{K_n(i)} K_n(A) \xrightarrow{K_n(\pi)} K_n(A/I) \tag{4}$$

is exact for every n.

Proof. As we have previously noted, it follows from their definition and Theorem 2.9 that proving the result for K_n' is equivalent to proving it for K_n. Moreover, by using the functoriality and exactness of S, one can see that the diagram

$$\begin{tikzcd}
K_n'(I) \arrow[r] \arrow[d] & K_n'(A) \arrow[r] \arrow[d] & K_n'(A/I) \arrow[d] \\
K_0(\mathrm{Im}(S^n i)) \arrow[r] & K_0(S^nA) \arrow[r] & K_0(S^nA/Im(S^ni))
\end{tikzcd}$$

is commutative and has isomorphisms as columns, for every $n \geq 0$. Thus, we only need to prove that $\ker(K_0(\pi)) \subset \mathrm{Im}(K_0(i))$, as we can restrict ourselves to $n = 0$ and the other inclusion is clear.

Now, given an element $[p]_0 - [s(p)]_0$ in $\ker(K_0(\pi))$, find a unitary $u \in U_N(\widetilde{A/I})$ such that, for suitable integers n, k, N, $u(\tilde{\pi} \oplus 1_n \oplus 0_k) = s(p) \oplus 1_n \oplus 0_k$. Then, by taking a unitary w homotopic to 1_{2N} in $U_{2N}(A)$ such that $\tilde{\pi}(w) = u \oplus u^*$, we can define the projection

$$r = w(p \oplus 1_n \oplus 0_{k+N})w^*.$$

As $\tilde{\pi}(r) \in M_\infty(\mathbb{C}1_A)$ by construction, it follows that $r \in M_\infty(\tilde{I})$. In particular, we have that

$$[p]_0 - [s(p)]_0 = [r]_0 - [s(r)]_0 \in \mathrm{Im}(\mathrm{K}_0(\mathrm{i})),$$

as required. □

Theorem 3.4. *For any exact sequence of the form (3), there exists a group homomorphism δ_1 such that the following sequence is exact:*

$$\begin{array}{ccccc} K_1(I) & \xrightarrow{K_1(i)} & K_1(A) & \xrightarrow{K_1(\pi)} & K_1(A/I) \\ & & & & \big\downarrow \delta_1 \\ K_0(A/I) & \xleftarrow{K_0(\pi)} & K_0(A) & \xleftarrow{K_0(i)} & K_0(I). \end{array} \tag{5}$$

Proof. Given an element $[u]_1 \in K_1(A/I)$ with $u \in U_k^+(\widetilde{A/I})$, we define its image through the index map δ_1 as

$$\delta_1([u]_1) = [w(1_n \oplus 0_k)w^*]_0 + [1_n \oplus 0_k]_0,$$

where $v \in U_k^+(\widetilde{A/I})$ is such that $u \oplus v \sim_h 1_{n+k}$ in $U_k^+(\widetilde{A/I})$, and w is a unitary lift of $u \oplus v$. It can be proven that δ_1 is indeed a well defined group homomorphism; see, for example, [5, Prop. 8.1.3].

Then, it follows from Proposition 3.3 that we only need to prove the equalities $\mathrm{Im}(\delta_1) = \ker(\mathrm{K}_0(\mathrm{i}))$ and $\ker(\delta_1) = \mathrm{Im}(\mathrm{K}_1(\pi))$. Moreover, note that the inclusions $\mathrm{Im}(\delta_1) \subseteq \ker(\mathrm{K}_0(\mathrm{i}))$ and $\mathrm{Im}(\mathrm{K}_1(\pi)) \subseteq \ker(\delta_1)$ are clear.

Thus, let $[u]_1 \in \ker(\delta_1)$ with $u \in U_m^+(\widetilde{A/I})$ for some m, and let w be a unitary lift of $u \oplus u^*$. As $\delta_1([u]_1) = 0$, we can find an integer k and a matrix $v \in M_{2(k+2m)}(\tilde{I})$ such that $vv^* = 1_{2n} - q \oplus 1_k \oplus 0_n$ and $v^*v = 1_{2n} - (1_m \oplus 0_m) \oplus (1_k \oplus 0_n)$, where $q = w(1_m \oplus 0_m)w^*$ and $n = k + 2m$.

By using the previous two equalities together with $vv^*v = v$, it is easy to check that $\tilde{\pi}(v) = 0_m \oplus X$ for some $X \in M_{2n-m}(\mathbb{C}1_{\tilde{A}})$. Therefore, there exists a complex $(2n+m) \times (2n+m)$ matrix U such that $\tilde{\pi}(qw \oplus v) = u \oplus U$. As $U \sim_h 1_{2n+m}$, it follows that

$$[u]_1 = [u]_1 + [U]_1 = K_1(\pi)([qw \oplus v]_1) \in \mathrm{Im}(\mathrm{K}_1(\pi)).$$

Now let $x = [p]_0 - [1_n \oplus 0_n]_0 \in \ker(K_0(i))$ with $p \in P_{2n}(\tilde{I})$. Then, there exists an integer $k \in \mathbb{N}$ for which $p \oplus 1_k \oplus 0_m \sim_0 (1_n \oplus 0_n) \oplus 1_k \oplus 0_m =: s_k$, with $m = 3(2n+k)$. Moreover, we can find a complex matrix P such that $Ps_kP^* = 1_{n+k} \oplus 0_{n+m}$ and, consequently, we have

$$q := P(p \oplus 1_k \oplus 0_m)P^* \sim_0 1_{n+k} \oplus 0_{n+m} =: d.$$

Take $w \in U_\infty^+(\tilde{A})$ such that $wqw^* = d$ and note that $\tilde{\pi}(w)$ commutes with d. It follows that $\tilde{\pi}(w) = a \oplus b$ for some $a \in U_{n+k}^+(\tilde{A})$. Finally, as w is a unitary lift of $a \oplus b$, we get

$$x = [q]_0 - [d]_0 = [wdw^*]_0 - [d]_0 = \delta_1(\tilde{\pi}(a)) \in \mathrm{Im}(\delta_1),$$

as desired. □

4. Bott periodicity and the hexagonal exact sequence

We begin this section by noting that $M_n(\widetilde{SA})$ can be identified with the set of continuous functions $f \in C(\mathbb{T}, M_n(\tilde{A}))$ such that $f(1) \in M_n(\mathbb{C}1_{\tilde{A}})$. In particular, we can write

$$U_n(\widetilde{SA}) = \{f \in C(\mathbb{T}, U_n(\tilde{A})) \mid f(1) \in M_n(\mathbb{C}1_{\tilde{A}})\}.$$

By using this identification, we can now define the Bott map. What follows is a combination of [4, Ch. 11] and [5, Ch. 9]:

Definition 4.1. Let A be a unital C^*-algebra and take $p \in P_n(A)$. We define $f_p \in U_n(\widetilde{SA})$ as the map from $\mathbb{T}$ to A such that $f_p(z) = zp + (1_n - p)$. The Bott map $\beta_A \colon K_0(A) \to K_1(SA)$ is then defined as $\beta_A([p]_0 - [q]_0) = [f_p f_q^*]_1$ for any element $[p]_0 - [q]_0$ in $K_0(A)$.

Remark 4.2. It can be proven that β_A is a well defined homomorphism; see, e.g., [4, § 11.1].

If A is not unital, we define the Bott map of A to be the only homomorphism for which the following diagram is commutative:

$$\begin{tikzcd}
0 \arrow[r] & K_0(A) \arrow[r] \arrow[d, "\beta_A"] & K_0(\tilde{A}) \arrow[r, shift left] \arrow[d, "\beta_{\tilde{A}}"] & K_0(\mathbb{C}) \arrow[r] \arrow[l, shift left] \arrow[d, "\beta_{\mathbb{C}}"] & 0 \\
0 \arrow[r] & K_1(SA) \arrow[r] & K_1(S\tilde{A}) \arrow[r, shift left] & K_1(S\mathbb{C}) \arrow[r] \arrow[l, shift left] & 0.
\end{tikzcd}$$

In particular, it follows that we only need to prove that the Bott map is an isomorphism in the unital case. Thus, we will assume from now on that A has a unit.

We will first prove that β_A is surjective. Let $GL_0(M_n(A))$ be the set of invertible $n \times n$ matrices that are homotopic to the identity. Then, define the following sets:

$$\begin{aligned}
\mathrm{Inv}_0^n &:= C(\mathbb{T}, GL_0(M_n(A))), \\
\mathrm{Pol}_m^n &:= \{f \in \mathrm{Inv}_0^n \mid f(z) = \sum_{i=0}^{m} a_i z^i,\ a_i \in M_n(A)\}, \\
\mathrm{Trig}_m^n &:= \{f \in \mathrm{Inv}_0^n \mid f(z) = \sum_{i=-m}^{m} a_i z^i,\ a_i \in M_n(A)\}.
\end{aligned}$$

Remark 4.3. One can check that $U_n(\widetilde{SA})$ is a subset of Inv_0^n for every n, and that, if two unitaries are homotopic in Inv_0^n, then they are also homotopic in $U_n(\widetilde{SA})$; see [4, § 11.2].

As we have already mentioned in the introduction, we omit the proof of the next lemma.

Lemma 4.4 ([5, Lem. 9.2.3–9.2.7])**.** *For every unital C^*-algebra A and every integer $n \in \mathbb{N}$, we have:*

(i) for every $f \in \mathrm{Inv}_0^n$, there exists an integer m and an element $h \in \mathrm{Trig}_m^n$ such that $f \sim_h h$ in Inv_0^n;

(ii) for every integer m there exists a continuous map μ_m^n from Pol_m^n to Pol_1^{mn+n} such that $\mu_m^n(f)$ is homotopic to $f \oplus 1_{mn}$ in Inv_0^n, for every f;

(iii) for any degree one polynomial $f \in \mathrm{Pol}_1^n$ there exists an element $\gamma(f)$ of the form f_p such that $f \sim_h \gamma(f)$ in Pol_1^n; moreover, the map $f \mapsto \gamma(f)$ is continuous.

Proposition 4.5. *The Bott map is surjective.*

Proof. Let $[f]_1 \in K_1(SA)$ with $f \in U_n(\widetilde{SA})$. By Lemma 4.4(1), we can find an element $h \in \mathrm{Trig}_m^n$ such that $f \sim_h h$ in Inv_0^n. As $z^{-N} \oplus 1_{M-1} \sim_h f^*_{1_N \oplus 0_{M-N}}$ in $U_M(\widetilde{SA})$ for every pair N, M such that $N \leq M$, we get $(hz^N)z^{-N} \oplus 1_M \sim_h (hz^N \oplus 1_M) f^*_{p_N}$ for every $N \leq M$, and where $p_N = 1_N \oplus 0_{M-N}$.

In particular, if N is large enough, hz^N is polynomial. Thus, for any such N, Lemma 4.4(2) ensures that we can find a degree one polynomial r such that $hz^N \oplus 1_t \sim_h r$ for some t. Moreover, it follows from Lemma 4.4(3) that there exists some element f_p homotopic to r.

By now adding some extra 1's in the diagonal, we have that $f \oplus 1_{M+nt} \sim_h f_{p \oplus 0_M} f^*_{p_N}$ and, consequently, that $\beta_A([p]_0 - [p_N]_0) = [f]_1$. □

We now prove that β_A is injective. Once again, we will omit the proof of the following lemma.

Lemma 4.6 ([5, § 9.1.2 & Lem. 9.2.10])**.** *For every unital C^*-algebra A and every integer $n \in \mathbb{N}$, we have:*

(i) the map $\pi\colon \{f_p \mid p \in P_n(A)\} \to P_n(A)$ sending an element f_p to p is continuous;

(ii) for any homotopy $f \mapsto f_t$ in Inv_0^n, there exists a positive integer N such that $f \mapsto f_t$ can be uniformly approximated by a homotopy $c \mapsto c_t$ in Trig_N^n that is piecewise linear. In particular, if $f_0, f_1 \in \mathrm{Trig}_N^n$, one can set $c_0 = f_0$ and $c_1 = f_1$.

Proposition 4.7. *The Bott map is injective.*

Proof. Let $[p]_0 - [q]_0 \in K_0(A)$ be such that $\beta_A([p]_0 - [q]_0) = 0$ or, equivalently, such that $[f_p f_q^*]_1 = 0$. Then, by possibly adding some zeros diagonally, we have that $f_p \sim_h f_q$. By Lemma 4.6(2), we can find a polynomial homotopy between $z^N f_p$ and $z^N f_q$ for some integer N. Moreover, once again adding some zeros and ones diagonally, and using Lemma 4.4(2,3), we obtain a homotopy $t \mapsto f_{p_t}$ such that $p_0 = p$ and $p_1 = q$. Finally, Lemma 4.6(1) gives a homotopy between p and q, from which we get that $[p]_0 = [q]_0$ and, therefore $[p]_0 - [q]_0 \in \ker(\beta_A)$. □

Combining Propositions 4.5 and 4.7 above, one gets the desired result:

Theorem 4.8. *For any C^*-algebra A, the Bott map β_A is an isomorphism between the groups $K_0(A)$ and $K_2(A)$. Consequently, all the K-groups $K_n(A)$ of even subindexes are isomorphic to $K_0(A)$, and those with odd subindexes are isomorphic to $K_1(A)$.*

Other proofs of this result are indeed possible, such as the recent one in [6]. In it, Voiculescu's almost commuting matrices are used to define a homomorphism $\alpha_A\colon K_1(SA) \to K_0(A)$, which is then shown, with the help of Atiyah's rotation trick, to be the mutual inverse of β_A (the author thanks the anonymous referee for this reference).

With Bott periodicity at hand, we can now construct the hexagonal exact sequence.

Theorem 4.9. *For any exact sequence of the form* (3)*, there exists a group homomorphism* δ_0 *such that the following hexagonal sequence is exact*

$$\begin{tikzcd}
K_1(I) \arrow[r, "K_1(i)"] & K_1(A) \arrow[r, "K_1(\pi)"] & K_1(A/I) \arrow[d, "\delta_1"] \\
K_0(A/I) \arrow[u, "\delta_0"] & K_0(A) \arrow[l, "K_0(\pi)"'] & K_0(I). \arrow[l, "K_0(i)"']
\end{tikzcd}$$

Proof. Given an exact sequence $0 \to I \to A \to A/I \to 0$, consider the suspended sequence

$$0 \to SI \to SA \to S(A/I) \to 0$$

and its corresponding index δ_1' from Theorem 3.4. Then, define δ_0 as the composition $\theta_{I,1}^{-1} \circ \delta_1' \circ \beta_{A/I}$, where $\theta_{I,1}^{-1}$ is the isomorphism from Theorem 2.9.

$$\begin{tikzcd}
K_2(A/I) \arrow[d, "\theta_{I,1}^{-1}\circ\delta_1'"] & & \\
K_1(I) \arrow[r, "K_1(i)"] & K_1(A) \arrow[r, "K_1(\pi)"] & K_1(A/I) \arrow[d, "\delta_1"] \\
K_0(A/I) \arrow[uu, bend left, "\beta_{A/I}"] & K_0(A) \arrow[l, "K_0(\pi)"'] & K_0(I) \arrow[l, "K_0(i)"']
\end{tikzcd}$$

By Theorem 3.4, we only need to prove that the sequence is exact at $K_1(I)$ and $K_0(A/I)$. To do this, simply note that the following diagram is commutative

$$\begin{tikzcd}
K_0(A) \arrow[r, "K_0(\pi)"] \arrow[d, leftrightarrow, "\beta_A"'] & K_0(A/I) \arrow[r, "\delta_0"] \arrow[d, leftrightarrow, "\beta_{A/I}"'] & K_1(I) \arrow[r, "K_1(i)"] \arrow[d, leftrightarrow, "\theta_I"] & K_1(A) \arrow[d, leftrightarrow, "\theta_A"] \\
K_1(SA) \arrow[r, "K_1(S\pi)"] & K_1(S(A/I)) \arrow[r, "\delta_1"] & K_0(SI) \arrow[r, "K_0(Si)"] & K_0(SA),
\end{tikzcd}$$

and that all of its columns are isomorphisms. As the second row is exact by Theorem 3.4, so is the first one. □

Example 4.10. Let H be an infinite dimensional separable Hilbert space and consider the Calkin algebra $Q(H) = B(H)/K(H)$, where $K(H)$ is the algebra of compact operators on H. Then, the exact sequence

$$0 \longrightarrow K(H) \xrightarrow{i} B(H) \xrightarrow{\pi} Q(H) \longrightarrow 0$$

induces, by Theorem 4.9, the hexagonal exact sequence

$$\begin{tikzcd}
K_1(K(H)) \arrow[r, "K_1(i)"] & K_1(B(H)) \arrow[r, "K_1(\pi)"] & K_1(Q(H)) \arrow[d, "\delta_1"] \\
K_0(Q(H)) \arrow[u, "\delta_0"] & K_0(B(H)) \arrow[l, "K_0(\pi)"'] & K_0(K(H)). \arrow[l, "K_0(i)"']
\end{tikzcd}$$

Moreover, recall from Example 2.7 that $K_0(B(H)) = K_1(B(H)) = 0$. Thus, δ_0 and δ_1 are isomorphisms.

By using that the K-groups are stable (see [4, Prop. 6.4.1 & Prop. 8.2.8]), one can also see that $K_0(K(H)) \cong K_0(\mathbb{C}) \cong \mathbb{Z}$ and that $K_1(K(H)) \cong K_1(\mathbb{C}) = 0$. Therefore, the K_0 and K_1 groups of the Calkin algebra are isomorphic to 0 and $\mathbb{Z}$ respectively.

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