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Lefschetz properties in algebra and geometry

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Resum *(CAT)*
La propietat dèbil de Lefschetz (WLP) té un paper important tant a l'àlgebra com a la geometria. A [3], Mezzetti, Miró-Roig i Ottaviani van demostrar que el fet que certs ideals Artinians de $\mathbb{K}[x_0, \dots, x_n]$ fallin la WLP té relació amb l'existència de projeccions de la varietat de Veronese que satisfan una equació de Laplace. Aquest vincle dóna lloc a la definició de sistema de Togliatti. En aquest article, enunciarem alguns resultats recents obtinguts sobre el tema. En particular exposem la classificació dels sistemes de Togliatti minimals i llisos generats per $2n+3$ monomis de grau $d \geq 10$ obtinguts a [8].

Abstract *(ENG)*
The weak Lefschetz property (WLP) plays an important role both in algebra and geometry. In [3], Mezzetti, Miró-Roig and Ottaviani found that the failure of the WLP for some particular Artinian ideals in $\mathbb{K}[x_0, \dots, x_n]$ is related with the existence of projections of the Veronese variety satisfying one Laplace equation. This relation gives rise to the definition of Togliatti system. In this note, we state some recent results on this topic. In particular, we expose the classification of minimal smooth Togliatti systems generated by $2n+3$ monomials of degree $d \geq 10$ obtained in [8].

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1. Introduction

The weak and strong Lefschetz properties on graded Artinian algebras have been an object of study along the last few decades. We say that a graded Artinian algebra $A = \bigoplus_i A_i$ has the strong Lefschetz property (SLP) if the multiplication by a dth power of a general linear form has maximal rank (i.e., $\times L^d \colon A_i \to A_{i+d}$ is either injective or surjective for every i). In particular, we say that A has the weak Lefschetz property (WLP) if the same occurs for $d = 1$. These properties have connections among different areas such as algebraic geometry, commutative algebra and combinatorics. Sometimes quite surprisingly, these connections give new approaches to other problems, which are a priori are unrelated.

The study of the Lefschetz properties started in 1980 with the work Stanley [9], which reached the following result:

Proposition 1.1. *Let $R = \mathbb{K}[x_0, \dots, x_n]$ be the polynomial ring in n variables, and $I = (x_0^{a_0}, \dots, x_n^{a_n}) \subset R$ be an Artinian monomial complete intersection. Let $L \in R_1$ be a general linear form. Then, for any positive integers d and i, the homomorphism $\times L^d \colon [R/I]_i \to [R/I]_{i+d}$ (induced by multiplication by L^d) has maximal rank.*

Afterwards, Watanabe [12] continued this research connecting the Lefschetz properties to the Sperner theory in combinatorics. Later more connections between the Lefschetz properties and vector bundles, line arrangements on the plane or the Fröberg conjecture have been discovered (see, for instance, [6, 7]). In this note, however, we will focus on another connection based on the so-called Togliatti systems.

In Mezzetti–Miró-Roig–Ottaviani [3], the authors related the failure of the weak Lefschetz property of Artinian ideals to the existence of projective varieties satisfying at least one Laplace equation. This connection (see Proposition 2.10) between a pure algebraic notion and a differential geometry concept gives rise to Togliatti systems (see Definition 2.12), an important family of Artinian ideals generated by homogeneous forms of the same degree d failing the WLP in degree $d-1$.

Let $\mathbb{K}$ be an algebraically closed field of characteristic 0 and $R = \mathbb{K}[x_0, \dots, x_n]$. Given an ideal $I \subset R$ generated by homogeneous forms of the same degree d, there is no difference between $[R/I]_i$ and R_i whenever $i \le d-1$. Therefore, the lowest possible degree in which R/I can fail the WLP is $d-1$. Thus, Togliatti systems are precisely Artinian ideals I failing the WLP in the very first possible degree: from $d-1$ to d. Equivalently in [3, Thm. 3.2] it was proved that Togliatti systems give rise to projections of the Veronese variety $V(n, d)$ satisfying at least one Laplace equation of order $d-1$.

After reviewing this connection, the rest of this note is devoted to study the classification of minimal (smooth) monomial Togliatti systems (see 2.12) and present some recent results found in [8]. The complete classification of Togliatti systems is still an open problem. However, if we restrict our attention to *monomial* Togliatti systems a lot of combinatorial tools emerge and the picture becomes clearer (see, for instance, [2, 3]). In [4] the complete classification of minimal smooth monomial Togliatti systems of quadrics and cubics, was given. This classification uses graph theory and other combinatoric tools in its proof, and cannot be easily generalized to classify all minimal smooth monomial Togliatti systems of degree $d \ge 4$. In order to overcome this difficulty and address the classification problem for an arbitrary degree $d \ge 4$, a new strategy was proposed in [2]. First of all, upper and lower bounds on the number of generators $\mu(I)$ of a minimal monomial Togliatti system are given. Namely, if I is a minimal Togliatti system in R generated by monomials of degree $d \ge 4$, then $2n+1 \le \mu(I) \le \binom{n+d-1}{n}$ where $n \ge 2$ and $d \ge 4$. A second step consists of classifying all smooth monomial Togliatti systems reaching this lower bound or close to it; see Remark 3.3. In [2], a complete classification of all minimal smooth Togliatti systems generated by $2n+1 \leq \mu(I) \leq 2n+2$ monomials, was achieved. In particular, it was shown that, except a few cases when $n = 2$, these Togliatti systems have a very peculiar shape. However, this shape is not directly generalizable to Togliatti systems with $\mu(I) \geq 2n+3$. Actually, even when $\mu(I) = 2n+3$ very different situations occur. For instance, there is no *smooth* minimal Togliatti system generated by $2n+3$ monomials of degree $d \geq 4$ whenever $n \geq 3$; see Proposition 3.4. Therefore we focus the study on minimal Togliatti systems in three variables generated by 7 monomials of degree $d \geq 4$. In [8] a complete classification of all the minimal Togliatti systems in $\mathbb{K}[x, y, z]$ generated by 7 monomials (see Theorem 3.7) was given. As a corollary, a complete classification of *smooth* minimal Togliatti systems in R generated by $2n + 3$ monomials was achieved. Section 3 is devoted to present and motivate these two recent results.

2. Preliminaries

This section is devoted to recall all the definitions related to the Lefschetz properties and Laplace equations, and a review of the connection between those two notions and the definition of Togliatti systems.

Definition 2.1. Let $I \subset R$ be an Artinian ideal and let us consider $A = R/I$ with the standard graduation $A = \bigoplus_{i=0}^{r} A_i$. Let $L \in R_1$ be a general linear form. Then:

(i) A has the strong Lefschetz property (SLP) if, for all positive integer d and for all $1 \leq i \leq r-d$, the homomorphism $\times L^d \colon [A]_i \to [A]_{i+d}$ has maximal rank;

(ii) A has the weak Lefschetz property (WLP) if, for all $1 \leq i \leq r-1$, the homomorphism $\times L \colon [A]_i \to [A]_{i+1}$ has maximal rank.

Remark 2.2. It is clear that having the SLP implies having the WLP, however, the converse is not true. For instance, it can be proved that $I = (x_0^2, x_1^3, x_2^5, x_0x_1, x_0x_2^2, x_1x_2^3, x_1^2x_2^2)$ has the WLP but fails the SLP in degrees 2 and 1, and also that $I = (x_0^3, x_1^3, x_2^3, (x_0+x_1+x_2)^3)$ has the WLP but fails the SLP in degrees 3 and 1.

In this note we will focus on the failure of the weak Lefschetz property for Artinian ideals. Let us first see some examples:

Example 2.3. (i) $I = (x^3, y^3, z^3, xyz)$ fails the WLP in degree 2.

(ii) $I = (x^4, y^4, z^4, t^4, xyzt)$ fails the WLP in degree 5.

(iii) By [5, Thm. 4.3], the ideals $I = (x_0^{n+1}, \dots, x_n^{n+1}, x_0 \dots, x_n)$ fail the WLP in degree $\binom{n+1}{2} - 1$.

Remark 2.4. Notice that the first ideal fails the WLP in the first non trivial place while the others fail later. This particularity will be studied in the sequel in more detail.

Even though an Artinian ideal I is expected to have the WLP, establishing this property for a concrete family of Artinian ideals can be a hard problem. For instance, Stanley [9] and Watanabe [12] proved that a *general* Artinian complete intersection has the WLP. However, to see wether *every* Artinian complete intersection with codimension ≥ 4 has the WLP remains an open problem.

In the last decades there have been established multiple connections between Lefschetz properties and other areas of mathematics, such as combinatorics, representation theory or geometry; see, for instance, [3, 9, 12]. In particular, Mezzetti–Miró-Roig–Ottaviani [3] connected the failure of the WLP with rational varieties satisfying Laplace equations. Let us recall some differential geometry definitions.

Definition 2.5. Let $X \subset \mathbb{P}^N$ be a rational variety of dimension n with parametrization $\Psi \colon \mathbb{P}^n \dashrightarrow X$, $(t_0 : \ldots : t_n) \mapsto (F_0(t_0, \ldots, t_n) : \ldots : F_N(t_0, \ldots, t_n))$. We call *s-th osculating vector space* on $x = \Psi(t_0 : \ldots : t_n)$ the vector space

$$T_x^{(s)}X := \left\langle \frac{\partial^s \Psi}{\partial_{t_0}^{k_0} \ldots \partial_{t_n}^{k_n}}(t_0 : \ldots : t_n) | k_0 + \cdots + k_n = s \right\rangle.$$

Finally, we call *s-th osculating projective space* on $x \in X$ the projectivization of the vector space above: $\mathbb{T}_x^{(s)}X := \mathbb{P}\left(T_x^{(s)}X\right)$.

Remark 2.6. There are $\binom{n+s}{s} - 1$ vectors $(k_0, \ldots, k_n)$ satisfying $k_0 + \cdots + k_n = s$. Then, in a general point $x \in X$, the *expected dimension* of $T_x^{(s)}X$ is $\binom{n+s}{s} - 1$. However, if there are linear dependencies among the partial derivatives of order s, this bound is not reached. In this case, Ψ satisfies a linear partial differential equation of order s. This motivates the following definition.

Definition 2.7. Let $X \subset \mathbb{P}^N$ be a rational projective variety of dimension n. We say that X *satisfy δ Laplace equations of order s* if

(i) for all smooth point $x \in X$ we have $\dim T_x^{(s)}X < \binom{n+s}{s} - 1$, and

(ii) for general $x \in X$, $\dim T_x^{(s)}X = \binom{n+s}{s} - 1 - \delta$.

Remark 2.8. If $N < \binom{n+s}{s} - 1$, then $T_x^{(s)}X$ is spanned by more vectors than the ambiance space. So, X trivially satisfies at least one Laplace equation of order s.

Let us now restrict our attention to Artinian ideals generated by homogeneous forms of the same degree d. To these ideals we can associate two different rational varieties:

Definition 2.9. Let $I = (F_1, \ldots, F_r) \subset R$ be an Artinian ideal generated by r forms of degree d. Let I^{-1} be the ideal generated by the inverse Macaulay system of I; see, for instance, [3, § 3]. Consider $\phi_{[I^{-1}]_d} \colon \mathbb{P}^n \dashrightarrow \mathbb{P}^{\binom{n+d}{d}-r-1}$ to be the rational map associated to $[I^{-1}]_d$ and $\phi_{I_d} \colon \mathbb{P}^n \to \mathbb{P}^{r-1}$ to be the morphism (I is Artinian) associated to I_d. We define $X_{n,[I^{-1}]_d} := \overline{\mathrm{Im}(\phi_{[I^{-1}]_d})}$, which is the projection of the d-th Veronese variety $V(n, d)$ from $\langle F_1, \ldots, F_r \rangle$, and $X_{n,I_d} := \mathrm{Im}(\phi_{I_d})$, which is the projection of $V(n, d)$ from $\langle [I^{-1}]_d \rangle$.

Finally, we can establish the following important relation.

Proposition 2.10 (Mezzetti–Miró-Roig–Ottaviani, [3, Thm. 3.2]). *Let $I \subset R$ be an Artinian ideal generated by r forms $F_1, \ldots, F_r$ of degree d. If $r \leq \binom{n+d-1}{n-1}$, then the following conditions are equivalent:*

(a) the ideal I fails the WLP in degree $d - 1$;

(b) the forms $F_1, \ldots, F_r$ become k−linearly dependent on a general hyperplane $H \subset \mathbb{P}^n$; and

(c) the n dimensional variety $X_{n,[I^{-1}]_d}$ satisfies at least one Laplace equation of order $d - 1$.

Remark 2.11. (i) The bound on the number of generators ensures the possible failure of the WLP is due to injectivity.

(ii) For $1 \leq i \leq d-2$, since $[R/I]_i \cong R_i$, the homomorphism $\times L\colon [R/I]_i \to [R/I]_{i+1}$ is injective. Hence, the first possible degree where $\times L$ can fail to have maximum rank is precisely $d-1$.

This result motivates the following definitions.

Definition 2.12. Let $I = (F_1, \dots, F_r) \subset R$ be an Artinian ideal generated by $r \leq \binom{n+d-1}{n-1}$ forms of degree d. We say that I is a *Togliatti system* if it satisfies any of the three equivalent conditions from Proposition 2.10. Moreover, we say that

(i) I is a *minimal Togliatti system* if for any $1 \leq s \leq r$ and $\{F_{i_1}, \dots, F_{i_s}\} \subset \{F_1, \dots, F_r\}$, $I' = (F_{i_1}, \dots, F_{i_s})$ is not a Togliatti system;

(ii) I is a *monomial Togliatti system* if I can be generated by monomials;

(iii) I is a *smooth Togliatti system* if $X_{n,[I^{-1}]_d}$ is a smooth variety.

Remark 2.13. (i) The name was given in honor to E. Togliatti who proved that the only smooth Togliatti system of cubics is $I = (x_0^3, x_1^3, x_2^3, x_0x_1x_2)$; see, for instance, [10, 11].

(ii) To address the classification of Togliatti systems it is crucial to investigate when they are minimal. In the next section we will focus on the minimality of a Togliatti system.

(iii) If I is generated by r monomials of degree d, then I^{-1} is generated by all the $\binom{n+d}{d} - r$ monomials of degree d which do not generate I. In particular, $X_{n,I}$ and $X_{n,[I^{-1}]_d}$ are two closely related toric varieties. These varieties carry a lot of combinatorial properties which ease their study as can be seen in [2, 3, 4, 8]. Henceforward, we will restrict our attention to (smooth) monomial Togliatti systems.

This preliminary section ends with a result that shows why dealing with monomial ideals simplifies the study of Lefschetz properties.

Proposition 2.14 (Migliore–Miró-Roig–Nagel, [5, Prop. 2.2])**.** *Let $I \subset R$ be an Artinian monomial ideal. Then R/I has the WLP if and only if $x_0 + \cdots + x_n$ is a Lefschetz element for R/I.*

Therefore, to check the WLP for monomial Artinian ideals there is no need to study the multiplication by a *general* linear form L, but only for the particular form $L = x_0 + \cdots + x_n$.

3. Classification of monomial Togliatti systems

In this section we will address the problem of classifying (smooth) monomial Togliatti systems. We will start by taking account of all the most recent results achieved, all of them, in the present decade. Finally we will establish two new results that enlarge the classification of (smooth) monomial Togliatti systems. Furthermore, these new results provide a huge amount of new examples of monomial Togliatti systems which are non trivial in the sense of Proposition 3.2.

In Michałek–Miró-Roig [4], the authors completely classified all smooth monomial Togliatti systems of quadrics and cubics using graph theory. However, the classification problem of (smooth) monomial

Togliatti systems generated by forms of degree $d \geq 4$ becomes much more involved and, up to now, a complete classification seems out of reach. Therefore, Mezzetti and Miró-Roig changed the strategy in [2] and focused on studying the number of generators of a (smooth) monomial Togliatti system. Moreover, they gave upper and lower bounds of this number of generators, and classified all the smooth monomial Togliatti systems near the lower bound. Before stating the main results of this note, let us fix some notation and review some motivational results from [2].

Definition 3.1. For every $n, d \in \mathbb{N}$, we denote by $\mathcal{T}(n, d)$ the set of all minimal monomial Togliatti systems, and by $\mathcal{T}^s(n, d)$ the set of all minimal smooth monomial Togliatti systems. Furthermore, we write

(i) $\mu(n, d) = \min\{\mu(I) | I \in \mathcal{T}(n, d)\}$,

(ii) $\mu^s(n, d) = \min\{\mu(I) | I \in \mathcal{T}^s(n, d)\}$,

where $\mu(I)$ stands for the minimal number of generators of an ideal $I \subset \mathbb{K}[x_0, \dots, x_n]$.

With this notation, we start establishing lower bounds for the quantities $\mu(n, d)$ and $\mu^s(n, d)$. The first result in this direction gives rise to a very important family of Togliatti systems:

Proposition 3.2. *Let $n, d \in \mathbb{N}$ and let m be a monomial of degree $d-1$. Then, the ideal $(x_0^d, \dots, x_n^d) + m(x_0, \dots, x_n)$ is a minimal monomial Togliatti system. These minimal monomial Togliatti systems are called* trivial Togliatti systems*.*

Proof. By Propositions 2.14 and 2.10, it is enough to restrict the generators of I to the hyperplane $x_0 + \dots + x_n$, and see that they become linearly dependent; see, for instance, [2, Rem. 3.8]. □

Remark 3.3. (i) Let I be trivial Togliatti system given by a monomial $m = x_0^{a_0} \cdots x_n^{a_n}$ of degree $d-1$. Then, the n dimensional variety $X_{n,[I^{-1}]_d}$ parametrized by $\Psi = \phi_{[I^{-1}]_d}$ (see Definition 2.9) satisfies a very simple Laplace equation: $\partial^{d-1}/\partial^{a_0} \cdots \partial^{a_n} \Psi = 0$.

(ii) Observe that a trivial Togliatti system I satisfies that $2n+1 \leq \mu(I) \leq 2n+2$. In fact, Mezzetti and Miró-Roig showed in [2] that for $n \geq 2$ and $d \geq 4$, $\mu(n, d) = \mu^s(n, d) = 2n+1$. Furthermore, they proved that, for $d \gg 4$, all minimal monomial smooth Togliatti systems of forms of degree d with $2n+1 \leq \mu(I) \leq 2n+2$ are trivial.

(iii) Of course, for minimal monomial Togliatti systems I with $\mu(I) \geq 2n+3$ we cannot expect them to be trivial. In particular, its study gives rise to geometrically more interesting examples.

The remaining of this section is devoted to study the classification of (smooth) monomial Togliatti systems generated by $2n+3$ forms of degree d. Let us start the discussion by stating the following result:

Proposition 3.4 (Mezzetti–Miró-Roig, [2, Prop. 4.4])**.** *Let $n \geq 3$ and $d \geq 4$. Then, there is no $I \in \mathcal{T}^s(n, d)$ with $\mu(I) = 2n+3$.*

This result significantly reduces the task of classifying all minimal smooth monomial Togliatti systems I generated by $2n+3$ forms of degree $d \geq 4$. Namely, it will be enough to consider the three variables case. Moreover, the hypothesis of Proposition 2.10 implies that a Togliatti system in $\mathbb{K}[x, y, z]$ generated by 7 forms of degree d must satisfy that $7 \leq \binom{2+d-1}{2-1} = d+1$.

In other words, our goal now is shifted to classify all minimal smooth monomial Togliatti systems in $\mathbb{K}[x, y, z]$ generated by forms of degree $d \geq 6$. To establish this classification we need to fix some notation.

Definition 3.5. Let us denote the ideal $T=(x^3,y^3,z^3,xyz)$ and the following sets of monomial ideals:

$$A=\{(y^3,y^2z,yz^2,z^3),(xy^2,xz^2,y^3,z^3),(x^2y,y^3,y^2z,z^3),(x^2z,y^3,y^2z,z^3),(xz^2,y^3,y^2z,z^3),\\ (xz^2,y^3,y^2z,yz^2),(x^2z,y^3,y^2z,yz^2),(xyz,xz^2,y^3,yz^2),(xy^2,xz^2,y^3,yz^2),(xyz,xz^2,y^3,y^2z),\\ (xy^2,xz^2,y^2z,yz^2),(x^2z,xy^2,y^2z,yz^2),(x^2z,xz^2,y^3,y^2z),(x^2z,xz^2,y^3,yz^2),(x^2y,xy^2,y^3,z^3),\\ (x^2z,xy^2,y^3,z^3),(x^2z,xyz,y^3,y^2z),(x^2z,xyz,y^3,yz^2),(x^2y,xz^2,y^3,y^2z),(x^2y,xz^2,y^3,yz^2),\\ (x^2z,xy^2,y^3,yz^2)\},$$

$$B=\{(x^3z,xy^2z,y^4,yz^3),(x^2yz,xz^3,y^4,y^3z),(x^2z^2,xy^2z,y^4,z^4),(x^2yz,y^4,y^2z^2,z^4)\},$$

and

$$C=\{(x^3yz,xy^2z^2,y^5,z^5),(x^2yz^2,xy^3z,y^5,z^5)\}.$$

Finally, for any $d\geq 1$ integer, let be $M(d):=\{x^ay^bz^c\mid d-1\geq a,b,c\geq 0,\ a+b+c=d\}$.

This definition gives rise to the first examples of minimal Togliatti systems generated by 7 monomials of degree $d\geq 6$ in three variables.

Proposition 3.6. *Let $d\geq 6$. Then any of the following ideals is a minimal Togliatti system:*

(i) both (a) $I=(x^d,y^d,z^d)+m(x^2,y^2,xz,yz)$ and (b) $I=(x^d,y^d,z^d)+m(x^2,y^2,xy,z^2)$, for every $m\in M(d-2)$;

(ii) $I=(x^d,y^d,z^d)+x^{d-3}J$, for any $J\in A$;

(iii) $I=(x^d,y^d,z^d)+mT$, for all $m\in M(d-3)$;

(iv) $I=(x^d,y^d,z^d)+x^{d-4}J$, for every $J\in B$;

(v) $I=(x^d,y^d,z^d)+x^{d-5}J$, for any $J\in C$.

Proof. It is a strightforward computation to show that restricting the generators of each type of ideal to the linear form $x+y+z$ they become linearly dependent. Then, all of them are Togliatti systems according to Proposition 2.10. On the other hand, none of these ideals contain a Togliatti system generated by either 5 or 6 monomials. Hence, they are minimal. □

Moreover, as we will see in the next results, *almost* all minimal Togliatti systems in three variables generated by 7 monomials of degree $d\geq 6$ are of one of the types mentioned in Proposition 3.6. These results can be seen as a natural generalization of those obtained in Remark 3.3. The first result classifies all minimal Togliatti systems in three variables generated by 7 monomials of degree $d\geq 10$.

Theorem 3.7 (Miró-Roig–Salat, [8, Thm. 3.8])**.** *Let $I\subset\mathbb{K}[x,y,z]$ be a minimal Togliatti system generated by 7 monomials of degree $d\geq 10$. Then, up to a permutation of the variables, one of the following cases hold:*

(i) there is $m\in M(d-2)$ such that either (a) $I=(x^d,y^d,z^d)+m(x^2,y^2,xz,yz)$ or (b) $I=(x^d,y^d,z^d)+m(x^2,y^2,xy,z^2)$;

(ii) there is $J\in A$ such that $I=(x^d,y^d,z^d)+x^{d-3}J$;

(iii) *there is* $m \in M(d-3)$ *such that* $I = (x^d, y^d, z^d) + mT$;

(iv) *there is* $J \in B$ *such that* $I = (x^d, y^d, z^d) + x^{d-4}J$;

(v) *there is* $J \in C$ *such that* $I = (x^d, y^d, z^d) + x^{d-5}J$.

From this result, and using combinatorial properties of toric varieties, a complete classification of smooth minimal Togliatti systems in $n+1$ variables generated by $2n+3$ monomials of degree $d \geq 10$ can be given.

Set $M^0(d) = \{x_0^a x_1^b x_2^c \mid a+b+c = d \text{ and } a, b, c \geq 1\}$ for any integer $d \geq 3$. Then we can establish the following theorem regarding *smooth* Togliatti systems.

Theorem 3.8 (Miró-Roig–Salat, [8, Thm. 3.9]). *Let* $I \subset \mathbb{K}[x_0, \dots, x_n]$ *be a smooth minimal monomial Togliatti system of forms of degree* $d \geq 10$. *Assume that* $\mu(I) = 2n+3$. *Then* $n = 2$ *and, up to permutation of the coordinates, one of the following cases holds:*

(i) $I = (x_0^d, x_1^d, x_2^d) + m(x_0^2, x_1^2, x_0x_2, x_1x_2)$ *with* $m \in M^0(d-2)$;

(ii) $I = (x_0^d, x_1^d, x_2^d) + m(x_0^2, x_1^2, x_0x_1, x_2^2)$ *with* $m \in M^0(d-2)$;

(iii) $I = (x_0^d, x_1^d, x_2^d) + m(x_0^3, x_1^3, x_2^3, x_0x_1, x_2)$ *with* $m \in M^0(d-3)$.

Remark 3.9. For $6 \leq d \leq 9$ there are other examples of minimal monomial Togliatti systems $I = (x^d, y^d, z^d) + J \subset \mathbb{K}[x, y, z]$, generated by 7 monomials, which are not covered by Theorem 3.7. Using Macaulay2 software [1], we computed all of these additional ideals J, up to permutation of the variables: for $d = 6$,

$$(x^5y, x^3z^3, x^2y^3z, y^5z), (x^5z, x^3y^3, x^2y^2z^2, y^5z), (x^3z^3, x^2y^4, x^2y^2z^2, y^5z), (x^5z, x^3y^3, xyz^4, y^5z),$$
$$(x^4z^2, x^3y^3, x^2y^2z^2, y^4z^2), (x^3z^3, x^2y^4, x^2y^2z^2, y^4z^2), (x^4z^2, x^3y^3, xyz^4, y^4z^2), (x^3y^3, x^3z^3, x^2y^2z^2, y^3z^3),$$
$$xy(x^4, x^2y^2, xyz^2, y^4), xy(x^3z, x^2y^2, xyz^2, y^3z), xy(x^2y^2, x^2z^2, xyz^2, y^2z^2), xy(x^2y^2, x^2z^2, xz^3, y^2z^2),$$
$$xy(x^4, xz^3, y^4, y^2z^2), xy(x^4, x^2y^2, y^4, z^4), xy(x^4, xyz^2, y^4, z^4), xy(x^3z, x^2y^2, y^3z, z^4),$$
$$xz(x^3z, x^2z^2, xyz^2, y^4), xz(x^2yz, x^2z^2, xyz^2, y^4), xz(x^3z, xy^2z, xyz^2, y^4), xz(x^3z, x^2yz, xz^3, y^4),$$
$$xz(x^3z, x^2z^2, xz^3, y^4), xz(x^3z, xy^2z, xz^3, y^4), xz(x^2y^2, x^2z^2, xy^3, y^3z), xz(x^2y^2, x^2z^2, xy^2z, y^3z),$$
$$xz(x^2z^2, xy^3, xy^2z, y^3z), xz(x^2y^2, x^2z^2, xy^3, y^4), xz(x^2y^2, x^2z^2, y^4, y^3z), xz(x^2y^2, x^2z^2, y^4, y^2z^2),$$
$$xz(x^3z, x^2yz, xy^2z, y^4), xz(x^3z, x^2yz, xyz^2, y^4), x(xy^4, xyz^3, xz^4, y^3z^2), x(x^4z, x^2y^3, xy^2z^2, y^5),$$
$$x(x^4z, xyz^3, y^5, y^3z^2), x(x^2z^3, xy^4, xy^2z^2, y^5), x(x^4z, x^2yz^2, y^5, y^2z^3), x(x^2z^3, xy^4, xyz^3, y^3z^2),$$
$$x(x^4z, x^2z^3, xy^3z, y^5), x(x^4z, x^2y^3, y^5, yz^4), x(x^3z^2, x^2y^3, xz^4, y^3z^2), x(x^4z, xy^2z^2, y^5, yz^4),$$
$$x(x^2z^3, xy^4, xz^4, y^3z^2), x(x^2y^3, x^2z^3, y^4z, yz^4), x(x^4y, x^2z^3, xy^3z, y^5), x(x^2yz^2, xy^3z, y^5, z^5);$$

for $d = 7$,

$$xy(x^2z^3, xy^4, xy^2z^2, y^5), xy(x^5, x^2y^2z, xyz^3, y^5), xy(x^4y, x^3y^2, xz^4, y^3z^2), xy(x^3y^2, x^2y^3, x^2z^3, y^2z^3),$$
$$xy(x^5, x^2y^2z, y^5, z^5), xy(x^4z, xy^4, y^5, z^5), xy(x^5, xyz^3, y^5, z^5), xz(x^3z^2, x^2z^3, xy^3z, y^5),$$
$$xz(x^4z, x^2yz^2, xz^4, y^5), xz(x^4z, xy^3z, xz^4, y^5), x(x^5z, x^2y^3z, xy^2z^3, y^6), x(xy^5, xy^2z^3, xz^5, y^4z^2),$$
$$x(x^5z, x^4y^2, x^2y^2z^2, y^3z^3), x(x^4y^2, x^4z^2, x^2y^2z^2, y^3z^3), x(x^3y^3, x^3z^3, x^2y^2z^2, y^3z^3),$$
$$x(x^4yz, x^2y^4, x^2z^4, y^3z^3), x(x^2y^4, x^2y^2z^2, x^2z^4, y^3z^3), x(x^4yz, xy^5, xz^5, y^3z^3), x(x^2y^2z^2, xy^5, xz^5, y^3z^3),$$
$$x(x^5z, x^2y^3z, y^6, yz^5), x(x^5z, xy^2z^3, y^6, yz^5), x(x^5z, x^4y^2, y^5z, yz^5), x(x^4y^2, x^4z^2, y^5z, yz^5),$$
$$x(x^3y^3, x^3z^3, y^5z, yz^5), x(x^4yz, x^2y^2z^2, y^6, z^6), x(x^4yz, y^6, y^3z^3, z^6), x(x^2y^2z^2, y^6, y^3z^3, z^6),$$
$$xyz(x^2y^2, x^2z^2, xy^3, y^4), xyz(x^3z, x^2yz, xy^2z, y^4), xyz(x^4, x^2y^2, xyz^2, y^4), xyz(x^3z, x^2yz, xyz^2, y^4),$$
$$xyz(x^3z, x^2z^2, xyz^2, y^4), xyz(x^2yz, x^2z^2, xyz^2, y^4), xyz(x^3z, xy^2z, xyz^2, y^4), xyz(x^3z, x^2yz, xz^3, y^4),$$

$$xyz(x^3z, x^2z^2, xz^3, y^4), xyz(x^3y, xy^3, xz^3, y^4), xyz(x^3z, xy^2z, xz^3, y^4), xyz(x^2y^2, x^2z^2, xy^3, y^3z),$$
$$xyz(x^2y^2, x^2z^2, xy^2z, y^3z), xyz(x^2z^2, xy^3, xy^2z, y^3z), xyz(x^3z, x^2y^2, xyz^2, y^3z), xyz(x^3y, x^2z^2, xyz^2, y^3z),$$
$$xyz(x^2y^2, x^2z^2, y^4, y^3z), xyz(x^4, xy^3, xz^3, y^2z^2), xyz(x^4, xz^3, y^4, yz^3);$$

for $d = 8$,

$$xy(x^4z^2, x^3y^3, xyz^4, y^4z^2), xz(x^3z^3, x^2y^2z^2, xy^4z, y^6);$$

and for $d = 9$,

$$xyz(x^3z^3, x^2y^2z^2, xy^4z, y^6), xyz(x^3y^3, x^3z^3, x^2y^2z^2, y^3z^3), xyz(x^6, x^2y^2z^2, y^6, z^6).$$

This remark added to Proposition 3.7 completes the classification problem, for any degree $d \geq 6$, of all minimal monomial Togliatti systems in three variables with 7 generators.

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