Compactification of a diagonal action on the product of CAT(-1) spaces

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Resum *(CAT)*

Sigui X un espai CAT(-1) propi i no compacte, i sigui Γ un subgrup discret cocompacte de les isometries de X. Compactifiquem l'acció diagonal de Γ a $X \times X$ considerant un domini de la frontera per horofuncions respecte a la mètrica del màxim.

Abstract *(ENG)*

Let X be a proper, non-compact CAT(-1) space, and Γ a discrete cocompact subgroup of the isometries of X. We compactify the diagonal action of Γ on $X \times X$ considering a domain of the horofunction boundary with respect to the maximum metric.

1. Introduction

Let (X, d_X) be a proper, non-compact CAT(-1) space. As examples of such spaces one can think of complete simply connected Riemannian manifolds of negative sectional curvature, for instance the hyperbolic space, and of metric trees. For a review on CAT(-1) spaces one can consult the first section of Bourdon [3]. Here, we consider a discrete and cocompact subgroup Γ of the isometries of (X, d_X), and the diagonal action of Γ on the product space $X \times X$,

$$
\begin{array}{rcl}
\Gamma \times X \times X & \longrightarrow & X \times X \\
(\gamma, x, y) & \mapsto & (\gamma x, \gamma y).
\end{array}
$$

This action is not cocompact but one can try to attach to $X \times X$ a set Ω of ideal boundary points such that the action on $X \times X \cup \Omega$ is cocompact.

The space $(X \times X, d_{X \times X})$, where $d_{X \times X}$ denotes the standard product metric, is a proper metric space and hence, it can be compactified by means of horofunctions; see, for instance, Ballmann–Gromov–Schroeder [2, Sec. 3]. Since it is a CAT(0) space, the horofunctions are, in fact, Busemann functions and the horofunction boundary coincides with the boundary by rays; see Ballmann [1, Prop. 2.5]. The action of Γ extends continuously to an action by homeomorphisms on the boundary of the compactification, but it is not clear if there is a subset of the boundary that can be a good candidate for Ω. For this reason we introduce the maximum metric in $X \times X$, defined by

$$
d_{\max}((x, y), (x', y')) = \max\{d_X(x, x'), d_X(y, y')\},
$$

for any (x, y), (x', y') in $X \times X$. The space $(X \times X, d_{\max})$ is also a proper metric space and therefore, it can also be compactified by horofunctions. However, it is not a CAT(0) space, since the geodesic segment joining two points is not unique. The group Γ acts on $(X \times X, d_{\max})$ by isometries and the action can be extended to an action by homeomorphisms on the ideal boundary, which we denote by $\partial_\infty^{\max}(X \times X)$. The compactification with respect to the metric $d_{\max}$ turns out to be more adapted to our problem. The main result of this work is the fact that we can find a subset of $\partial_\infty^{\max}(X \times X)$ where the action of Γ is properly discontinuous, and which compactifies the action of Γ on $X \times X$:

Theorem 1.1. *Let X be a proper, non-compact CAT(-1) space and Γ a group of isometries of X acting in a properly discontinuous and cocompact way on X. There exist an open set $\Omega \subset \partial_\infty^{\max}(X \times X)$ such that the diagonal action of Γ on $X \times X \cup \Omega$ is properly discontinuous and cocompact.*

The ideal boundary $\partial_\infty^{\max}(X \times X)$ can be interpreted in terms of the ideal boundary of X, which we denote by $\partial_\infty X$. We will see that $\partial_\infty^{\max}(X \times X)$ splits in a singular part, which is naturally homeomorphic to $\partial_\infty X \sqcup \partial_\infty X$, and a regular part, which is homeomorphic, also in a natural way, to $\partial_\infty X \times \partial_\infty X \times \mathbb{R}$. If we denote by D the diagonal of $\partial_\infty X \times \partial_\infty X$, the set Ω is just the subset of the regular part of the boundary which corresponds, under the homeomorphism, to the set $((\partial_\infty X \times \partial_\infty X) \setminus D) \times \mathbb{R}$.

The set Ω is naturally homeomorphic to the set G of parametrized geodesics in X, equipped with the topology of uniform convergence on compact sets. The identification gives more geometrical insight to the solution of the problem. We consider the diagonal Δ of $X \times X$, and the nearest point retraction that sends each point of $X \times X$ to its nearest point in Δ. This map can be extended in a continuous way to G, by sending each geodesic g to the point $(g(0), g(0))$ in Δ. We use the continuous extension of the nearest point retraction to show that the action of Γ on Ω is properly discontinuous and cocompact.

2. Definitions and notations

In this section we review very briefly the main concepts that appear through the paper.

For us, a parametrized geodesic (or simply a geodesic) is an isometric embedding $g \colon \mathbb{R} \to X$. We call the image of a geodesic a *geodesic line*. A *ray* is an isometric embedding $r \colon [0, \infty) \to X$. And similarly, a *geodesic segment* joining two points x and y is an isometric embedding $xy \colon [a, b] \to X$ such that $xy(a) = x$ and $xy(b) = y$. We will make no distinction between a ray or a geodesic segment and their images.

A *geodesic space* is a metric space such that any two points can be joined by a geodesic segment. A metric space is *proper* if its closed balls are compact. A geodesic metric space is proper if and only if it is complete and locally compact; see Gromov [7, Thm. 1.10].

Let X be a metric space and Δ a geodesic triangle in X. A comparison triangle $\bar{\Delta}_{\mathbb{H}^2}$ in the hyperbolic plane $\mathbb{H}^2$ (or $\bar{\Delta}_{\mathbb{E}^2}$ in the Euclidean plane $\mathbb{E}^2$) is a geodesic triangle in $\mathbb{H}^2$ (or in $\mathbb{E}^2$) with sides of the same length than those of Δ. The space X is *CAT*(-1) (respectively, *CAT*(0)) if for any triangle Δ, any x, y in Δ and their comparison points $\bar{x}, \bar{y}$ in $\bar{\Delta}_{\mathbb{H}^2}$ (respectively, $\bar{\Delta}_{\mathbb{E}^2}$) satisfy $d(x, y) \leq d(\bar{x}, \bar{y})$. A CAT($-1$) space is, in particular, CAT(0); see Bridson–Haefliger [4, Part II, Thm. 1.12].

The action of Γ on X is *properly discontinuous* if, for every compact $K \subset X$, the set $K \cap \gamma K$ is non empty for finitely many $\gamma \in \Gamma$. And it is *cocompact* if there exists a compact $K \subset X$ such that $X = \Gamma K$.

Given a metric space (X, d), the *Gromov product* of two points x and y in X with respect to a third point z in X is defined as

$$(x|y)_z = \frac{1}{2}\big(d(z, y) + d(z, x) - d(x, y)\big).$$

In a CAT(-1) space the product can be extended to points at infinity. Observe that a geodesic g determines two rays, one with the same orientation than the geodesic itself and the other with the reversed orientation. The equivalence classes of these rays are the ideal endpoints of the geodesic, we denote them by $g(+\infty)$ and $g(-\infty)$ respectively. The Gromov product of the points $g(+\infty)$ and $g(-\infty)$ with respect to a base point o, is given by

$$(g(+\infty)|g(-\infty))_o = \frac{1}{2}\big(\beta^{g(0)}_{g(+\infty)}(o) + \beta^{g(0)}_{g(-\infty)}(o)\big),$$

where the functions $\beta^{g(0)}_{g(+\infty)}$, $\beta^{g(0)}_{g(-\infty)}$ will be defined in Section 3. The definition of the Gromov product is independent from the parametrization of the geodesic line L with ideal endpoints $g(+\infty)$ and $g(-\infty)$. Observe that if $o \in L$ then it is 0. See, for instance, Ghys–de-la-Harpe [6, Ch. 2] and Bourdon [3, Sect. 2], for more about Gromov products.

3. Horofunction boundary of $(\mathbf{X} \times \mathbf{X}, \mathbf{d_{max}})$

As we have stated in the introduction, let (X, d_X) be a proper, non-compact CAT(-1) space and consider the product space $X \times X$ together with the maximum metric $d_{\max}$,

$$d_{\max}((x, y), (x', y')) = \max\{d_X(x, x'), d_X(y, y')\}.$$

The compactification via horofunctions of a proper metric space is explained in detail in Ballmann [1, Ch.2] and Bridson–Haefliger [4, Part II, Sect.8]. The idea, which is in fact valid for any complete locally

compact metric space, is due to Gromov [2, Sect.3]. It consists on embedding the space, in our case $(X \times X, d_{\max})$, into the space C_* of its continuous functions (with the topology of uniform convergence on compact sets) modulo additive constant, via the map

$$\iota \colon X \times X \longrightarrow C_*$$
$$(x, y) \longmapsto [d_{\max}((x, y), \cdot)],$$

assigning to each point in the space the class in C_* of the distance function with respect to this point. The closure of the space, denoted by $\overline{X \times X}^{\max}$, is the closure of $\iota(X \times X)$ in C_*, and the ideal boundary, denoted by $\partial_\infty^{\max}(X \times X)$, is the set $(\overline{X \times X}^{\max}) \setminus \iota(X \times X)$. Both $\overline{X \times X}^{\max}$ and $\partial_\infty^{\max}(X \times X)$ are compact since $X \times X$ is locally compact.

A horofunction is a continuous function such that its class belongs to $\partial_\infty^{\max}(X \times X)$. One can think of horofunctions as limits of normalized distance functions. The level sets of a horofunction are known as horospheres and the sublevel sets as horoballs. Observe that two horofunctions in the same equivalence class differ by a constant and share the same set of horospheres and horoballs.

We can compactify in the same way the original space (X, d_X), since it is also a proper metric space. In this case, since (X, d_X) is CAT(-1), the compactification obtained is homeomorphic to the compactification by rays, where the points at infinity are equivalence classes of rays, and two rays c, c' are in the same class if $d_X(c(t), c'(t)) \leq C$ for all t. The equivalence class of a ray c is denoted by $c(\infty)$, the ideal boundary of X by $\partial_\infty X$, and an arbitrary point in $\partial_\infty X$ by ξ. The horofunctions of a CAT(0) space are in fact Busemann functions and can be written as $\beta_\xi^p(\cdot) = \lim_{t \to \infty} d_X(\cdot, c(t)) - t$, where c is a ray such that $c(0) = p$ and $c(\infty) = \xi \in \partial_\infty X$. Observe that β_ξ^p is the representative of $\xi \in \partial_\infty X$ that satisfies $\beta_\xi^p(p) = 0$. The following two properties about Busemann functions will be used:

(i) If a sequence $\{x_n\}_n$ converges to a point ξ in $\partial_\infty X$, then $\beta_\xi^o(y) = \lim_{n \to \infty} d(x_n, y) - d(x_n, o)$; see Ballmann [1, Prop. 2.5].

(ii) $\beta_\xi^o(y) = -\beta_\xi^y(o)$; see Bourdon [3, Sect.2.1].

Example 3.1. Let X be a complete simply connected n-dimensional Riemannian manifold X of sectional curvature ≤ -1, then $\partial_\infty X \cong S^{n-1}$. In particular, $\partial_\infty \mathbb{H}^n \cong S^{n-1} \cong (T_{x_0}\mathbb{H}^n)^1$ for any $x_0 \in X$.

Now we list all the possible boundary points obtained when compactifying $(X \times X, d_{\max})$. For the calculations, we have chosen a base point $O = (o, o)$ with $o \in X$ and then, as a representative of a class of distance functions, the function $d_{\max}^O((x, y), \cdot) = d_{\max}((x, y), \cdot) - d_{\max}((x, y), (o, o))$. Then, $\{[d_{\max}(P_n, \cdot)]\}_n \to \xi$ if and only if $\{d_{\max}^O(P_n, \cdot)\}_n \to h_\xi^O$, where h_ξ^O is the horofunction in ξ satisfying $h_\xi^O(O) = 0$.

Proposition 3.2. *Every divergent sequence $\{(x_n, y_n)\}_n \subset (X \times X, d_{\max})$ has a subsequence that satisfies, except for permutations of x_n and y_n, one of the following possibilities:*

(i) *$d_X(x_n, o)$ is bounded for all n, and $\{y_n\}_n \to \xi' \in \partial_\infty X$;*

(ii) *$\{x_n\}_n \to \xi \in \partial_\infty X$, $\{y_n\}_n \to \xi' \in \partial_\infty X$, and $d_X(x_n, o) - d_X(y_n, o) \to C$;*

(iii) *$\{x_n\}_n \to \xi \in \partial_\infty X$, $\{y_n\}_n \to \xi' \in \partial_\infty X$, and $d_X(x_n, o) - d_X(y_n, o) \to +\infty$.*

Furthermore, for every $(z, z') \in X \times X$, the limit $\lim_{n \to \infty} d^O_{\max}((x_n, y_n), (z, z'))$ equals $\beta^o_\xi(z')$ for the possibility (i), $\max\{\beta^o_\xi(z), \beta^o_{\xi'}(z') + C\}$ for (ii), and $\beta^o_\xi(z)$ for (iii).

Proof. We do the proof for case (II), the other two cases can be obtained in a similar fashion. Denote $C_n = d(x_n, o) - d(y_n, o)$ and assume $C_n \geq 0$, to simplify. Then,

$$d_{\max}((x_n, y_n), (x, y)) - d_{\max}((x_n, y_n), (o, o)) = \max\{d(x_n, x) - d(x_n, o), d(y_n, y) - d(y_n, o) - C_n\}.$$

Given $\epsilon > 0$ and $r > 0$, we want to see that there is an N such that, for all $n > N$ and all $(x, y) \in B^{\max}(O, r)$,

$$|\max\{d(x_n, x) - d(x_n, o), d(y_n, y) - d(y_n, o) - C_n\} - \max\{\beta^o_\xi(x), \beta^o_{\xi'}(y) - C\}| < \epsilon,$$

where $B^{\max}(O, r)$ is the ball of center $O = (o, o)$ and radius r in $(X \times X, d_{\max})$. There are four cases to check. We do the case for which the first maximum is $d(x_n, x) - d(x_n, o)$ and the second maximum is $\beta^o_{\xi'}(y) - C$, the other are similar. Because of the definition of Busemann function, there is an N such that, for all $(x, y) \in B^{\max}(O, r)$ and all $n > N$,

$$d(x_n, x) - d(x_n, o) - \beta^o_\xi(x) < \epsilon,$$

$$d(y_n, y) - d(y_n, o) - C_n - (\beta^o_{\xi'}(y') - C) > -\epsilon.$$

On the one hand, we have

$$d(x_n, x) - d(x_n, o) - (\beta^o_{\xi'}(y) - C) = d(x_n, x) - d(x_n, o) - \beta^o_\xi(x) + \beta^o_\xi(x) - (\beta^o_{\xi'}(y) - C) < \epsilon$$

and

$$d(x_n, x) - d(x_n, o) - (\beta^o_{\xi'}(y) - C) = d(x_n, x) - d(x_n, o) - (d(y_n, y) - d(y_n, o) - C_n) +$$
$$d(y_n, y) - d(y_n, o) - C_n - (\beta^o_{\xi'}(y') - C) > -\epsilon. \qquad \square$$

The set of boundary points with a representative of the form $\beta^o_\xi(z)$ or $\beta^o_{\xi'}(z')$ is called the *singular part* of the boundary, denoted $\partial^{\max}_\infty(X \times X)_{\text{sing}}$. The rest of the points, i.e., those with a representative of the form $\max\{\beta^o_\xi(z), \beta^o_{\xi'}(z') + C\}$, form the *regular part* of the boundary, denoted by $\partial^{\max}_\infty(X \times X)_{\text{reg}}$.

Observe that, for any constant C', the function $\max\{\beta^o_\xi(z), \beta^o_{\xi'}(z') + C\} + C' = \max\{\beta^o_\xi(z) + C', \beta^o_{\xi'}(z') + C + C'\}$ is in the same class as $\max\{\beta^o_\xi(z), \beta^o_{\xi'}(z') + C\}$. Since two Busemann functions of X associated to the same point $\xi \in \partial_\infty X$ differ by a constant, for each C' we can find points p and p' in X such that $\beta^p_\xi(z) = \beta^o_\xi(z) + C'$ and $\beta^{p'}_{\xi'}(z') = \beta^o_{\xi'}(z') + C + C'$. So, the regular points are in fact the classes modulo constant of the functions $\max\{\beta^p_\xi(z), \beta^{p'}_{\xi'}(z')\}$ for all $p, p' \in X$ and $\xi, \xi' \in \partial_\infty X$.

Proposition 3.3. *There is a natural homeomorphism $\varphi_{sing} \colon \partial^{\max}_\infty(X \times X)_{sing} \longrightarrow \partial_\infty X \sqcup \partial_\infty X$ that consists in associating to a Busemann function that takes values only in the first (second) factor of $X \times X$, the same Busemann function viewed as a point of the first (second) factor in $\partial_\infty X \sqcup \partial_\infty X$.*

Proof. It follows from the fact that the set of Busemann functions in one factor is naturally identified to the boundary of X. $\qquad \square$

The regular part of the boundary can also be identified with a more easy to handle object.

Proposition 3.4. *For each choice of base point $(o, o') \in X \times X$, there is a natural homeomorphism*

$$\varphi_{reg} \colon \partial_\infty^{\max}(X \times X)_{reg} \longrightarrow \partial_\infty X \times \partial_\infty X \times \mathbb{R}$$

$$\left[\max\{\beta_\xi^p(z), \beta_{\xi'}^{p'}(z')\}\right] \mapsto (\xi, \xi', \beta_{\xi'}^{p'}(o) - \beta_\xi^p(o')). \tag{1}$$

Remark 3.5. Under our choice of base point, the homeomorphism (1) takes the form

$$\varphi_{reg} \colon \partial_\infty^{\max}(X \times X)_{reg} \longrightarrow \partial_\infty X \times \partial_\infty X \times \mathbb{R}$$

$$\max\{\beta_\xi^o(z), \beta_{\xi'}^o(z') + C\} \mapsto (\xi, \xi', C). \tag{2}$$

Proof of Proposition 3.4. The map (1) is well defined since two horofunctions in the same class differ by a constant, and $\max\{\beta_\xi^p, \beta_{\xi'}^{p'}\} \neq \max\{\beta_\eta^q, \beta_{\eta'}^{q'}\}$ for $(\xi, \xi') \neq (\eta, \eta')$. To see this, normalize the Busemann functions with respect to the same point; i.e., $\max\{\beta_\xi^p, \beta_{\xi'}^{p'}\} = \max\{\beta_\xi^p, \beta_{\xi'}^p + A\}$ and $\max\{\beta_\eta^q, \beta_{\eta'}^{q'}\} = \max\{\beta_\eta^p + B, \beta_{\eta'}^p + C\}$ for some constants A, B, C. Choose a sequence $z_n \to \xi$ along the geodesic ray joining p and ξ. Then $\beta_\xi^p(z_n) \to -\infty$ and $\beta_\nu^p(z_n) \to +\infty$ for all $\nu \in \partial_\infty X$ such that $\nu \neq \xi$. Using this property one can see that $\xi = \eta$ and $\xi' = \eta'$ if $\max\{\beta_\xi^p, \beta_{\xi'}^{p'}\} = \max\{\beta_\eta^q, \beta_{\eta'}^{q'}\}$.

Now, for each class we choose the representative of the form $\max\{\beta_\xi^o(z), \beta_{\xi'}^o(z') + C\}$ and prove that the map (2) is a homeomorphism. Injectivity is clear. For the exhaustivity, given (ξ, ξ', C), one can see that the sequence $(g(n), g(-n))$ where g is a parameterization of the geodesic line joining ξ' and ξ with $\beta_\xi^o(g(0)) - \beta_{\xi'}^o(g(0)) = C$, has limit $\max\{\beta_\xi^o, \beta_{\xi'}^o + C\}$.

For the continuity, take a sequence $\max\{\beta_{\xi_n}^o, \beta_{\xi_n'}^o + C_n\}$ converging to a point $\max\{\beta_\xi^o, \beta_{\xi'}^o + C\}$. The C_n must be bounded, otherwise the sequence would converge to a Busemann function in one factor. This and the compactness of the set of Busemann fuctions of X, implies that (ξ_n, ξ_n', C_n) converges to some point (η, η', C'). Since the maximum function is continuous, $\max\{\beta_{\xi_n}^o, \beta_{\xi_n'}^o + C_n\}$ should also converge to $\max\{\beta_\eta^o, \beta_{\eta'}^o + C'\}$. Therefore, $\max\{\beta_\xi^o, \beta_{\xi'}^o + C\} = \max\{\beta_\eta^o, \beta_{\eta'}^o + C'\}$ and $(\xi, \xi', C) = (\eta, \eta', C')$.

The continuity of the maximum function also assures that the inverse of (2) is continuous. $\square$

Example 3.6. For X a complete simply connected n-dimensional Riemannian manifold of sectional curvature ≤ -1, $\partial_\infty^{\max}(X \times X)_{reg} \cong S^{n-1} \times S^{n-1} \times \mathbb{R}$ and $\partial_\infty^{\max}(X \times X)_{sing} \cong S^{n-1} \sqcup S^{n-1}$. It can be shown that the boundary of $X \times X$ is homeomorphic to a $(2n-1)$-sphere:

$$\partial_\infty^{\max}(X \times X) \cong \mathrm{Join}(S^{n-1}, S^{n-1}) \cong S^{2n-1}.$$

4. An ideal domain for the action of Γ

From now on, Γ will be a discrete and cocompact subgroup of the isometries of X. Recall that we were interested in the diagonal action of Γ on $X \times X$. In this section we look for an open subset of $\partial_\infty^{\max}(X \times X)$, where the action of Γ is good enough. Let D denote the diagonal in $\partial_\infty X \times \partial_\infty X$,

$$D = \{(\xi, \xi) \mid \xi \in \partial_\infty X\} \cong \partial_\infty X,$$

and let $\Lambda_{(x,y)}$ be the limit set of the orbit of the point (x, y) in $X \times X$,

$$\Lambda_{(x,y)} = \overline{\Gamma(x, y)} \cap \partial_\infty^{\max}(X \times X).$$

Notice that $\Lambda_{(x,y)}$ depends on $(x,y) \in X \times X$ since this space is not CAT(-1). Define the limit set of Γ as

$$\Lambda_\Gamma = \bigcup_{(x,y) \in X \times X} \Lambda_{(x,y)}.$$

Proposition 4.1. *The diagonal action of Γ on $X \times X$ satisfies: (i) $\Lambda_\Gamma \subset \partial_\infty^{\max}(X \times X)_{reg}$; and (ii) $\varphi_{reg}(\Lambda_\Gamma) = D \times \mathbb{R}$.*

Proof. First observe that the limit of any sequence $(\gamma_n x, \gamma_n y)$ is in $D \times \mathbb{R}$. Indeed, by the triangle inequality, $|d(\gamma_n x, o) - d(\gamma_n y, o)| \le d(x,y)$ so, the limit is a regular point and, since $d(\gamma_n x, \gamma_n y) = d(x,y)$, if $\gamma_n x \to \xi$ then $\gamma_n y \to \xi$ because X is CAT(-1).

Let us see next that any point (ξ, ξ, C) is in the limit set. Let γ_n be a sequence in Γ such that $\gamma_n \to \xi$. Observe that such a sequence exists since Γ is cocompact and hence its limit set in $\overline{X}$ is the whole $\partial_\infty X$. Let $\xi' \in \partial_\infty X$ be the limit of the sequence γ_n^{-1} and take any point (x,y) satisfying $\beta_{\xi'}^o(x) - \beta_{\xi'}^o(y) = C$. For instance, one can take a point $(g(t), g(t'))$, where g is the ray joining o and ξ', and $t - t' = C$. Then

$$
\begin{aligned}
\lim_{n \to \infty} d(\gamma_n x, o) - d(\gamma_n y, o) &= \lim_{n \to \infty} d(x, \gamma_n^{-1} o) - d(\gamma_n^{-1} o, o) - (d(y, \gamma_n^{-1} o) - d(\gamma_n^{-1} o, o)) \\
&= \beta_{\xi'}^o(x) - \beta_{\xi'}^o(y) = C.
\end{aligned}
$$

Hence, the limit of the sequence $(\gamma_n x, \gamma_n y)$ is the point (ξ, ξ, C). $\square$

As a candidate for the domain at infinity, we choose $\Omega \subset \partial_\infty^{\max}(X \times X)_{reg}$ such that $\Omega \cong (\partial_\infty X \times \partial_\infty X \setminus D) \times \mathbb{R}$ under the homeomorphism (1). Observe that we have excluded the whole region $D \times \mathbb{R}$.

Now consider the set G of parameterized geodesics in X. This set is the same as the set of oriented geodesic lines with a distinguished base point and, as we show next, it is in correspondence with the points of Ω. Observe that there is a natural action of Γ on G: an element $\gamma \in \Gamma$ sends a geodesic g to γg.

Lemma 4.2. *The following map is a bijection*

$$
\begin{aligned}
f : G &\longrightarrow \Omega \\
g &\mapsto \lim_{n \to \infty}(g(n), g(-n)).
\end{aligned}
$$

Proof. First we check that, given a geodesic g in G, the limit of the sequence $\{(g(n), g(-n))\}_n$ belongs to Ω. In order for this limit to be a regular point of the boundary, the sequence $\{(g(n), g(-n))\}_n$ needs to belong to case (ii) of Proposition 3.2 so, the limit of the difference $d_X(g(n), o) - d_X(g(-n), o)$ has to be a real constant. This follows from the next calculation:

$$
\begin{aligned}
\lim_{n \to \infty} \Big(d_X(g(n), o) - d_X(g(-n), o) \Big) &= \lim_{n \to \infty} \Big(d_X(g(n), o) - d_X(g(n), g(0)) \\
&\qquad + d_X(g(-n), g(0)) - d_X(g(-n), o) \Big) \\
&= \beta_{g(+\infty)}^{g(0)}(o) + \beta_{g(-\infty)}^{g(0)}(o) \\
&= \beta_{g(-\infty)}^o(g(0)) - \beta_{g(+\infty)}^o(g(0)) \in \mathbb{R}.
\end{aligned}
$$

Here, we have used the fact that $d_X(g(n), g(0)) = d_X(g(-n), g(0))$ and the definition of Busemann function. Henceforth, the limit of the sequence $\{(g(n), g(-n))\}_n$ is the point

$$\left(g(+\infty), g(-\infty), \beta_{g(-\infty)}^o(g(0)) - \beta_{g(+\infty)}^o(g(0)) \right)$$

in $\partial_\infty X \times \partial_\infty X \times \mathbb{R}$ under the homeomorphism (2). Since $g(+\infty) \neq g(-\infty)$, this limit belongs to Ω.

Let us check the injectivity of the map f. Suppose we have two different geodesics g and g' such that $f(g) = f(g')$. Then, $g(+\infty) = g'(+\infty)$ and $g(-\infty) = g'(-\infty)$ and, since given two ideal points in a CAT(-1) space there is a unique geodesic line having them as ideal endpoints (see Bridson–Haefliger [4, Thm. 9.33]), both geodesics must be different parametrizations of the same geodesic line L. Now, observe that the difference $\beta^o_{g(-\infty)}(g(0)) - \beta^o_{g(+\infty)}(g(0))$ can be rewritten using the Gromov product $(g(+\infty)|g(-\infty))_o$ as

$$
\begin{aligned}
\beta^o_{g(-\infty)}(g(0)) - \beta^o_{g(+\infty)}(g(0)) &= \beta^{g(0)}_{g(+\infty)}(o) - \beta^{g(0)}_{g(-\infty)}(o)) \\
&= 2(\beta^{g(0)}_{g(+\infty)}(o) - (g(+\infty)|g(-\infty))_o) \\
&= -2(\beta^o_{g(+\infty)}(g(0)) + (g(+\infty)|g(-\infty))_o).
\end{aligned}
$$

Therefore, the two parametrizations satisfy

$$
-2(\beta^o_{g(+\infty)}(g(0)) + (g(+\infty)|g(-\infty))_o) = -2(\beta^o_{g'(+\infty)}(g'(0)) + (g'(+\infty)|g'(-\infty))_o)
$$

and hence, $\beta^o_\xi(g(0)) = \beta^o_\xi(g'(0))$. Since both $g(0)$ and $g'(0)$ belong to L, we must have $g(0) = g'(0)$ so, both parametrizations of L do coincide.

To finish, let us check the exhaustivity of f. Let (ξ_+, ξ_-, r) be a point in Ω (seen through the homeomorphism (1)). We are looking for a geodesic g such that $\lim_{n\to\infty}\{(g(n), g(-n))\}_n = (\xi_+, \xi_-, r)$. We have already calculated the limit of such a sequence at the beginning of the proof, and we know it is the point $(g(+\infty), g(-\infty), \beta^o_{g(-\infty)}(g(0)) - \beta^o_{g(+\infty)}(g(0)))$. So, we look for a geodesic such that $g(+\infty) = \xi_+$, $g(-\infty) = \xi_-$, and $\beta^o_{g(-\infty)}(g(0)) - \beta^o_{g(+\infty)}(g(0)) = r$. Let L be the geodesic line with ideal endpoints ξ_+ and ξ_-. Consider a parametrization $g(t)$ of L. Since the functions $\beta^o_{\xi_+}(g(t))$ and $\beta^o_{\xi_-}(g(t))$ are lineal with slope ± 1, respectively, there is a unique point p in L satisfying $\beta^o_{\xi_-}(p) - \beta^o_{\xi_+}(p) = r$. The parametrization $g'(t)$ of L such that $g'(0) = p$ is the one we are looking for, since it satisfies $g'(+\infty) = \xi_+$, $g'(-\infty) = \xi_-$, and $\beta^o_{g'(-\infty)}(g'(0)) - \beta^o_{g'(+\infty)}(g'(0)) = r$. $\square$

Remark 4.3. Observe that what we have checked in the proof of Lemma 4.2 is, in fact, the bijectivity of the map $\varphi_{reg} \circ f$.

We consider in G the topology of uniform convergence on compact sets.

Theorem 4.4. *The following map is an equivariant homeomorphism*

$$
\begin{aligned}
f : G &\longrightarrow \Omega \\
g &\mapsto \lim_{n\to\infty}(g(n), g(-n)).
\end{aligned}
$$

Proof. To see that f is a homeomorphism, consider the map

$$
\begin{aligned}
f' : G &\longrightarrow ((\partial_\infty X \times \partial_\infty X) \setminus D) \times \mathbb{R} \\
g &\mapsto \left(g(+\infty), g(-\infty), \beta^{g(0)}_{g(+\infty)}(o) - \beta^{g(0)}_{g(-\infty)}(o)\right),
\end{aligned}
$$

which is just $\varphi_{reg} \circ f$ and which, as we have seen along the proof of Lemma 4.2, it is a bijection. Let ϕ be the Hopf parametrization

$$
\begin{aligned}
\phi : G &\longrightarrow ((\partial_\infty X \times \partial_\infty X) \setminus D) \times \mathbb{R}) \\
g &\mapsto \left(g(+\infty), g(-\infty), \beta^{g(0)}_{g(+\infty)}(o)\right),
\end{aligned}
$$

which is a homeomorphism (see Bourdon [3, Sect. 2.9]), and let h be the map

$$h\colon ((\partial_\infty X \times \partial_\infty X) \setminus D) \times \mathbb{R}) \longrightarrow ((\partial_\infty X \times \partial_\infty X) \setminus D) \times \mathbb{R})$$
$$(\xi_+, \xi_-, r) \longmapsto (\xi_+, \xi_-, 2(r - (\xi_+|\xi_-)_o)),$$

which is also a homeomorphism. Now, the following diagram commutes

$$
\begin{array}{ccc}
G & \xrightarrow{\quad f' \quad} & ((\partial_\infty X \times \partial_\infty X) \setminus D) \times \mathbb{R}). \\
\downarrow{\scriptstyle \phi} & \nearrow{\scriptstyle h} & \\
((\partial_\infty X \times \partial_\infty X) \setminus D) \times \mathbb{R} & &
\end{array}
$$

Indeed, for any $g \in G$,

$$
\begin{aligned}
h \circ \phi(g) &= h((g(+\infty), g(-\infty), \beta^{g(0)}_{g(+\infty)}(o))) \\
&= (g(+\infty), g(-\infty), 2(\beta^{g(0)}_{g(+\infty)}(o) - (g(+\infty)|g(-\infty))_o)) \\
&= (g(+\infty), g(-\infty), \beta^{g(0)}_{g(+\infty)}(o) - \beta^{g(0)}_{g(-\infty)}(o)) \\
&= f'(g).
\end{aligned}
$$

Since both ϕ and g are homeomorphisms, f' is a homeomorphism. And since $f' = \varphi_{reg} \circ f$ and ϕ_{reg} is a homeomorphism, our map f is a homeomorphism too.

To finish, observe that the map f is equivariant since, for every $\gamma \in \Gamma$ and $g \in G$,

$$f(\gamma g) = \lim_{n \to \infty} (\gamma g(n), \gamma g(-n)) = \gamma \lim_{n \to \infty} (g(n), g(-n)) = \gamma f(g). \qquad \square$$

The next proposition is a consequence of Theorem 4.4 and the properties of G. It will also be a consequence of Theorem 5.9.

Proposition 4.5. *The action of Γ on Ω is cocompact and properly discontinuous.* $\qquad \square$

5. Compactness of $(X \times X \cup \Omega)/\Gamma$

We define a topology in $X \times X \cup G$ in the following way. We keep the same topology in $X \times X$ and G, and we say a sequence $\{(x_n, y_n)\}_n$ in $X \times X$ converges to a point g in G if and only if the following three conditions hold: (i) $\{x_n\}_n \to g(+\infty)$, (ii) $\{y_n\}_n \to g(-\infty)$; and (iii) $d_X(x_n, o) - d_X(y_n, o) \to \beta^o_{g(-\infty)}(g(0)) - \beta^o_{g(+\infty)}(g(0))$. With this topology, $X \times X \cup G$ and $X \times X \cup \Omega$ are homeomorphic.

Now, consider the diagonal in $X \times X$, $\Delta = \{(x, x) \mid x \in X\}$, and let $\rho\colon X \times X \to \Delta$ be the map that sends each point in $X \times X$ to its nearest point in Δ with respect to the metric $d_{\max}$.

Lemma 5.1. *The fibre $\rho^{-1}(a, a)$ is the set of point $(x, y) \in X \times X$ such that a is the midpoint of the segment xy.* $\qquad \square$

Definition 5.2. We extend the projection ρ to a map $\tilde{\rho}\colon X \times X \cup G \to \Delta$ as follows: if $p = (x, y) \in X \times X$, then $\tilde{\rho}(p) = \rho(p)$; if $g \in G$, then $\tilde{\rho}(g) = (g(0), g(0))$.

Next, we prove that this extension is continuous. Before, we need a couple of auxiliary lemmas.

Lemma 5.3. *In a proper CAT(-1) space, $\lim_{i,j}(x_i|y_j)_o = (\xi|\xi')_o$ for any sequences $x_i \to \xi$, $y_j \to \xi'$; see Buyalo–Schroeder [5, Prop 3.4.2]. Moreover, $(\xi|\xi')_o = +\infty$ if and only if $\xi = \xi'$; see Bridson–Haefliger [4, Ch.III.H. Rmk. 3.17].* $\qquad\square$

Lemma 5.4. *Given a sequence of geodesic segments $g_n \colon [a_n, b_n] \to X$ such that $g_n(a_n) \to \xi$, $g_n(b_n) \to \xi'$, and $g_n(0) \to m \in X$, there exists a convergent subsequence to a geodesic g satisfying $g(+\infty) = \xi$, $g(-\infty) = \xi'$, and $g(0) = m$.*

Proof. The fact that the sequence $\{g_n\}_n$ converges to a geodesic g such that $g(0) = m$ is a consequence of Arzela–Ascoli theorem for proper metric spaces; see Papadopoulos [8, Thm. 1.4.9]. Now, for any $t \in \mathbb{R}$, observe that $(g_n(a_n)|g_n(b_n))_{g_n(t)} = 0$ since, for all n, $g_n(t)$ is a point of the segment g_n, and $\lim_n(g_n(a_n)|g_n(b_n))_{g_n(t)} = (\xi|\xi')_{g(t)}$ by the continuity of the Gromov product. Therefore, for all t, $(\xi|\xi')_{g(t)} = 0$ and $g(t)$ belongs to the line joining ξ and ξ'. Since this line is unique, $g(+\infty) = \xi$ and $g(-\infty) = \xi'$. $\qquad\square$

Proposition 5.5. *The map $\tilde{\rho} \colon X \times X \cup G \to \Delta$ is continuous.*

Proof. The restrictions of $\tilde{\rho}$ to $X \times X$ and to G are continuous. Let $\{(x_n, y_n)\}_n$ be a sequence in $X \times X$ converging to a geodesic g in G so, $x_n \to g(+\infty)$, $y_n \to g(-\infty)$, and $d(x_n, o) - d(y_n, o) \to C$. First, we will prove that the geodesic segments $x_n y_n$ converge to a parameterization of the geodesic line L with ideal endpoints $\xi = g(+\infty)$ and $\xi' = g(-\infty)$; then we will see that this parameterization is precisely g.

For each pair (x_n, y_n) let m_n be the middle point of the segment $x_n y_n$. The points m_n lie in a compact set. Indeed, suppose $d(o, m_n) \to +\infty$ so, $m_n \to \eta \in \partial_\infty X$. Using the definition of Gromov product,

$$(x_n|m_n)_o = \frac{1}{2}\big((x_n|y_n)_o + d(m_n, o) + \frac{1}{2}(d(x_n, o) - d(y_n, o))\big).$$

Since $\xi \neq \xi'$, by Lemma 5.3 $\lim_n(x_n|y_n)_o$ is bounded. By hypothesis, $d(x_n, o) - d(y_n, o)$ is also bounded and $d(m_n, o) \to +\infty$. Therefore, $\lim_n(x_n|m_n)_o = +\infty$ and, by Lemma 5.3 again, $\eta = \xi$. Similarly, one could find that $\eta = \xi'$, so $\xi = \xi'$ and arrive to a contradiction. Therefore, $m_n \to m$ for some $m \in X$ and, by Lemma 5.4, m must be a point in L.

Now we have, on one hand,

$$d_X(m_n, y_n) - d_X(o, y_n) - (d_X(m_n, x_n) - d_X(o, x_n)) = d_X(o, x_n) - d_X(o, y_n) \to C.$$

On the other hand, since $d_X(\cdot, y_n) - d_X(o, y_n) \to \beta^o_{\xi'}(\cdot)$ and $d_X(\cdot, x_n) - d_X(o, x_n) \to \beta^o_\xi(\cdot)$ uniformly on compact sets, we have

$$d_X(m_n, y_n) - d_X(o, y_n) - (d_X(m_n, x_n) - d_X(o, x_n)) \to \beta^o_{\xi'}(m) - \beta^o_\xi(m)$$

so, $\beta^o_{\xi'}(m) - \beta^o_\xi(m) = C$. But the only point in L satisfying this equation is precisely $g(0)$. Hence $m = g(0)$.

Therefore, $\tilde{\rho}(x_n, y_n) = (m_n, m_n) \to (m, m) = (g(0), g(0)) = \tilde{\rho}(g)$ and the map $\tilde{\rho}$ is everywhere continuous. $\qquad\square$

Corollary 5.6. *The fibre $\tilde{\rho}^{-1}(x, x)$ restricted to G is the set $G_x = \{\sigma \in G \mid \sigma(0) = x\} \subset G$.* $\qquad\square$

Example 5.7. In a Riemannian manifold X of dimension n and sectional curvature ≤ -1, G_x is identified with the unitary tangent at x, $(T_x X)^1 \cong S^{n-1}$.

Observe that, since $X \times X \cup G$ and $X \times X \cup \Omega$ are homeomorphic, we also have a continuous projection from $X \times X \cup \Omega$ to Δ, which we also call $\tilde{\rho}$.

Now, let $K \subset X$ be a compact in X such that $K/\Gamma \cong X/\Gamma$ and consider $K_\Delta = \{(x,x) \in \Delta \mid x \in K\}$.

Lemma 5.8. *The set $\tilde{\rho}^{-1}(K_\Delta)$ is compact.*

Proof. Consider a sequence $\{(x_n, y_n)\}_n$ in $\tilde{\rho}^{-1}(K_\Delta)$. Then, since K_Δ is compact, $\{\tilde{\rho}((x_n, y_n))\}_n$ has a convergent subsequence $\{(m_n, m_n)\}_n$ in K_Δ. Take a sequence of points $\{(x_n, y_n)\}_n$ that project to this convergent subsequence. They have a subsequence $\{(x'_n, y'_n)\}_n$ converging in $\overline{X \times X}^{\max}$. If the limit point of this subsequence is in $X \times X \cup \Omega$ we are done. If not, either the limit is in $X \times X \cup \Delta$ or $|\{d(x'_n, o) - d(y'_n, o)\}_n|$ is unbounded for all n. For the first case, observe that, for every n, $(x'_n|y'_n)_o \leq d_X(m_n, o) < C$, but $\{x'_n\}_n$ and $\{y'_n\}_n$ have the same limit if and only if $(x'_n|y'_n)_o \to \infty$ by Lemma 5.3, hence, this case is not possible. The second case is not possible either since $|\{d(x'_n, o) - d(y'_n, o)\}_n|$ unbounded implies $\{d(m_n, o)\}_n$ unbounded.

Now, consider a sequence of the form $\{g_n\}_n$ in $\tilde{\rho}^{-1}(K_\Delta)$. The geodesics in the sequence satisfy $g_n(0) \in K$ for all n. Since the set of geodesics going through a compact set is compact, $\{g_n\}$ has a convergent subsequence. □

Theorem 5.9. *The action of Γ on $X \times X \cup \Omega$ is properly discontinuous and cocompact.*

Proof. In order to see that Γ acts properly discontinuously, take $K \subset (X \times X \cup \Omega)$ any compact subset, and let $\gamma \in \Gamma$ be such that $\gamma K \cap K \neq \emptyset$. Then, $\tilde{\rho}(K) \cap \tilde{\rho}(\gamma K) = \tilde{\rho}(K) \cap \gamma \tilde{\rho}(K) \neq \emptyset$. Since $\tilde{\rho}$ is continuous, $\tilde{\rho}(K)$ is compact and, since the action of Γ on Δ is properly discontinuous, $\tilde{\rho}(K) \cap \gamma \tilde{\rho}(K) \neq \emptyset$ only for a finite number of elements $\gamma \in \Gamma$. Therefore $\gamma K \cap K \neq \emptyset$ only for a finite number of $\gamma \in \Gamma$.

For the cocompactness observe that, by Lemma 5.1 and Corollary 5.6, $(X \times X \cup \Omega)/\Gamma \cong (X \times X \cup G)/\Gamma = \tilde{\rho}^{-1}(K_\Delta)/\Gamma$ which is compact by Lemma 5.8. □

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