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Some characterizations of Howson PC-groups

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Resum *(CAT)*
Es demostra que, a la classe dels grups parcialment commutatius, les condicions d'ésser Howson, ésser totalment residualment lliure, i ésser producte lliure de grups lliure-abelians, són equivalents.

Abstract *(ENG)*
We show that, within the class of partially commutative groups, the conditions of being Howson, being fully residually free, and being a free product of free-abelian groups, are equivalent.

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In [10], the authors study the family of finitely generated partially commutative groups for which the fixed points subgroup of every endomorphism is finitely generated. Concretely, they characterize this family as those groups consisting in (finite) free products of finitely generated free-abelian groups.

In this note we provide an elementary proof for two extra characterizations of this same family, namely: being Howson, and being a limit group. Moreover, we observe that, for some of the properties, no restriction in the cardinal of the generating set is needed, and the result holds in full generality (i.e. for every — possibly infinitely generated — partially commutative group).

1. Preliminaries

We call *partially commutative groups* (*PC-groups*, for short) the groups that admit a presentation in which all the relations are commutators between generators, i.e. a presentation of the form $\langle X \mid R \rangle$, where R is a subset of $[X, X]$ (the *set* of commutators between elements in X).

We can represent this situation in a very natural way through the (simple) graph $\Gamma = (X, E)$ having as vertices the generators in X, and two vertices $x, y \in X$ being adjacent if and only if its commutator $[x, y]$ belongs to R; then we say that the PC-group is *presented* by the graph Γ, and we denote it by $\langle \Gamma \rangle$. Recall that a *simple graph* is undirected, loopless, and without multiple edges; so, Γ is nothing more than a symmetric and irreflexive binary relation in X.

A subgraph of a graph $\Gamma = (X, E)$ is said to be *full* if it has exactly the edges that appear in Γ over the same vertex set, say $Y \subseteq X$. Then, it is called the *full subgraph of* Γ *spanned by* Y, and we denote it by $\Gamma[Y]$. If Γ has a full subgraph isomorphic to a certain graph Λ, we will abuse the terminology and say that Λ is (or appears as) a full subgraph of Γ; we denote this situation by $\Lambda \leqslant \Gamma$. When none of the graphs belonging to a certain family $\mathcal{F}$ appear as a full subgraph of Γ, we say that Γ is $\mathcal{F}$*-free*. In particular, a graph Γ is Λ*-free* if it does not have any full subgraph isomorphic to Λ.

It is clear that every graph Γ presents exactly one PC-group; that is, we have a surjective map $\Gamma \mapsto \langle \Gamma \rangle$ between (isomorphic classes of) simple graphs and (isomorphic classes of) PC-groups. A key result proved by Droms in [5] states that this map is, in fact, bijective. Therefore, we have an absolutely transparent geometric characterization of isomorphic classes of PC-groups: we can identify them with simple graphs.

This way, the PC-group corresponding to a graph with no edges is a free group, and the one corresponding to a complete graph is a free-abelian group (in both cases, with rank equal to the number of vertices). So, we can think of PC-groups as a generalization of these two extreme cases including all the intermediate commutativity situations between them.

Similarly, disjoint unions and *joins of graphs* (i.e. disjoint unions with all possible edges between distinct constituents added) correspond to free products and weak direct products of PC-groups, respectively. So, for example, the finitely generated free-abelian times free group $\mathbb{Z}^m \times F_n$ is presented by the join of a complete graph of order m and an edgeless graph of order n.

All these facts are direct from definitions, and make the equivalence between the conditions in the following lemma almost immediate as well.

Lemma 1.1. *Let Γ be an arbitrary simple graph, and $\langle \Gamma \rangle$ the corresponding PC-group. Then, the following conditions are equivalent:*

(i) the path on three vertices P_3 is not a full subgraph of Γ (i.e. Γ is P_3-free),

(ii) the reflexive closure of Γ is a transitive binary relation,

(iii) Γ *is a disjoint union of complete graphs,*

(iv) $\langle\, \Gamma \,\rangle$ *is a free product of free-abelian groups.* $\square$

The next lemma, for which we provide an elementary proof, is also well known. We will use it in the proof of Theorem 2.1.

Lemma 1.2. *Let* Γ *be an arbitrary simple graph, and* Y *a subset of vertices of* Γ*. Then, the subgroup of* $\langle\, \Gamma \,\rangle$ *generated by* Y *is isomorphic to the PC-group presented by* $\Gamma[Y]$*.*

Proof. Let X be the set of vertices of Γ (then $Y \subseteq X$), and consider the following two homomorphisms:

$$\begin{array}{ccc} \langle\, \Gamma[Y] \,\rangle & \xrightarrow{\alpha} & \langle\, \Gamma \,\rangle \\ y & \longmapsto & y \end{array} \quad , \qquad \begin{array}{rcc} \langle\, \Gamma \,\rangle & \xrightarrow{\rho} & \langle\, \Gamma[Y] \,\rangle \\ Y \ni y & \longmapsto & y \\ X \setminus Y \ni x & \longmapsto & 1 \end{array} .$$

It is clear that both α and ρ are well defined homomorphisms (they obviously respect relations). Moreover, note that the composition $\alpha\rho$ (α followed by ρ) is the identity map on $\langle\, \Gamma[Y] \,\rangle$. Therefore, α is a monomorphism, and thus $\langle\, \Gamma[Y] \,\rangle$ is isomorphic to its image under α, which is exactly the subgroup of $\langle\, \Gamma \,\rangle$ generated by Y, as we wanted to prove. $\square$

A group is said to satisfy the *Howson property* (or to be *Howson*, for short) if the intersection of any two finitely generated subgroups is again finitely generated. It is well known that free and free-abelian groups are Howson (see, for example, [1] and [6] respectively).

However, not every PC-group is Howson: for example, a free-abelian times free group (studied in [4]) turns out to be Howson if and only if it does not have $\mathbb{Z} \times F_2$ as a subgroup. So, it is a natural question to ask for a characterization of Howson PC-groups, and we will see in Theorem 2.1 that the very same condition (not containing $\mathbb{Z} \times F_2$ as a subgroup) works for a general PC-group.

For limit groups there are lots of different equivalent definitions. We shall use the one using fully residual freeness (see [12] for details): a group G is *fully residually free* if for every finite subset $S \subseteq G$ such that $1 \notin S$, there exist an homomorphism φ from G to a free group such that $1 \notin \varphi(S)$. Then, a *limit group* is a finitely generated fully residually free group. From this definition, it is not difficult to see that both free and free-abelian groups are fully residually free, and that subgroups and free products of fully residually free groups are again fully residually free.

2. Characterizations

As proved by Rodaro, Silva, and Sykiotis [10, Theorem 3.1], if we restrict to finitely generated PC-groups, Lemma 1.1 describes exactly the family of those having finitely generated fixed point subgroup for every endomorphism (or equivalently, those having finitely generated periodic point subgroup for every endomorphism).

In the following theorem, we provide two extra characterizations for the PC-groups described in Lemma 1.1 (including the infinitely generated case). For completeness in the description, we summarize them in a single statement together with the conditions discussed above.

Theorem 2.1. *Let Γ be an arbitrary (possibly infinite) simple graph, and $\langle\, \Gamma\, \rangle$ the PC-group presented by Γ. Then, the following conditions are equivalent:*

(a) $\langle\, \Gamma\, \rangle$ is fully residually free,

(b) $\langle\, \Gamma\, \rangle$ is Howson,

(c) $\langle\, \Gamma\, \rangle$ does not contain $\mathbb{Z} \times F_2$ as a subgroup,

(d) $\langle\, \Gamma\, \rangle$ is a free product of free-abelian groups.

Moreover, if Γ is finite, then the following additional conditions are also equivalent:

(e) For every $\varphi \in \operatorname{End} \langle\, \Gamma\, \rangle$, the subgroup $\operatorname{Fix} \varphi = \{g \in \langle\, \Gamma\, \rangle \;:\; \varphi(g) = g\}$ of fixed points of φ is finitely generated.

(f) For every $\varphi \in \operatorname{End} \langle\, \Gamma\, \rangle$, the subgroup $\operatorname{Per} \varphi = \{g \in \langle\, \Gamma\, \rangle \;:\; \exists n \geq 1\ \varphi^n(g) = g\}$ of periodic points of φ is finitely generated.

Proof. $(a) \Rightarrow (b)$. Dahmani obtained this result for limit groups (i.e. assuming $\langle\, \Gamma\, \rangle$ finitely generated) as a consequence of them being hyperbolic relative to their maximal abelian non-cyclic subgroups (see [3, Corollary 0.4]). We note that the finitely generated condition is superfluous for this implication since the Howson property involves only finitely generated subgroups, and every subgroup of a fully residually free group is again fully residually free.

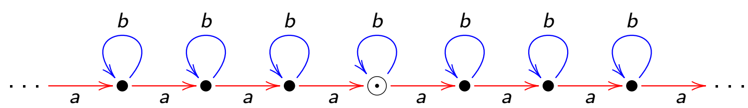
$(b) \Rightarrow (c)$. It is enough to prove that the group $\mathbb{Z} \times F_2$ does not satisfy the Howson property. The following argument is described as a solution to exercise 23.8(3) in [1] (see also [4]). Indeed, if we write $\mathbb{Z} \times F_2 = \langle\, t \mid -\, \rangle \times \langle\, a, b \mid -\, \rangle$, then the subgroups

$$\begin{aligned} H &= \langle a, b \rangle = F_2 \leqslant \mathbb{Z} \times F_2, \text{ and} \\ K &= \langle ta, b \rangle = \{ w(ta, b) \mid w \in F_2 \} = \{ t^{|w|_a} w(a, b) \mid w \in F_2 \} \leqslant \mathbb{Z} \times F_2 \end{aligned}$$

are both finitely generated, but its intersection

$$H \cap K = \{ t^0 w(a, b) \mid w \in F_2,\ |w|_a = 0 \} = \langle\!\langle b \rangle\!\rangle_{F_2} = \langle a^{-k} b a^k,\ k \in \mathbb{Z} \rangle$$

is infinitely generated, as you can see immediately from its Stallings graph



(see [11] and [8]), or using this alternative argument: suppose $H \cap K$ is finitely generated, then there exists $m \in \mathbb{N}$ such that $a^{m+1} b a^{-(m+1)} \in \langle a^{-k} b a^k,\ k \in [-m, m] \rangle$, and thus a^{m+1} equals the reduced form of some prefix of $w(a^m b a^{-m}, \ldots, b, \ldots, a^{-m} b a^m)$, for some word w. However, the sum of exponents of a in any such prefix must be in $[-m, m]$, which is a contradiction.

Note that both H and K are free groups of rank two whose intersection is infinitely generated. This fact, far from violating the Howson property of free groups, means that both H and K are not simultaneously contained in any free subgroup of $\mathbb{Z} \times F_2$.

$(c) \Rightarrow (d)$. From Lemma 1.2, if $\langle\, \Gamma \,\rangle$ does not contain the group $\mathbb{Z} \times F_2$ (which is presented by P_3) as a subgroup, then P_3 is not a full subgraph of Γ. Equivalently, $\langle\, \Gamma \,\rangle$ is a free product of free-abelian groups (see Lemma 1.1).

$(d) \Rightarrow (a)$. This is again clear, since free-abelian groups are fully residually free, and free products of fully residually free groups are again fully residually free. Note here, that no cardinal restriction is needed; neither for the rank of the free-abelian groups, nor for the number of factors in the free product, since the definition of fully residually freeness involves only finite families.

Finally, for the equivalence between (d), (e) and (f) under the finite generation hypothesis, see [10, Theorem 3.1]. $\square$

Observe that an immediate corollary of Lemma 1.2 is that the PC-group presented by any full subgraph $\Lambda \leqslant \Gamma$ is itself a subgroup of the PC-group presented by Γ, i.e. for every pair of graphs Γ, Λ,

$$\Lambda \leqslant \Gamma \;\Rightarrow\; \langle\, \Lambda \,\rangle \leqslant \langle\, \Gamma \,\rangle\,.$$

This property provides a distinguished family of subgroups (which we will call visible) of any given PC-group. More precisely, we will say that a PC-group $\langle\, \Lambda \,\rangle$ is a *visible subgroup of* a PC-group $\langle\, \Gamma \,\rangle$ — or that $\langle\, \Lambda \,\rangle$ is *visible in* $\langle\, \Gamma \,\rangle$ — if Λ appears as a full subgraph of Γ.

Of course, visible subgroups are PC-groups as well, but not every partially commutative subgroup of a PC-group is visible (for example, F_3 is obviously not visible in F_2).

Note that although "visibility" is a relative property (a PC-group can be visible in a certain group, and not in another one), there exist PC-groups which are visible in every PC-group in which they appear as a subgroup; we will call them *explicit*. That is, a given PC-group $\langle\, \Lambda \,\rangle$ (or the graph Λ presenting it) is *explicit* if for every graph Γ,

$$\Lambda \leqslant \Gamma \;\Leftrightarrow\; \langle\, \Lambda \,\rangle \leqslant \langle\, \Gamma \,\rangle\,.$$

For example, it is straightforward to see that the only explicit edgeless graphs are the ones with zero, one, and two vertices: the first two cases are obvious, and for the third one, note that if $F_2 \leqslant G$ then G can not be abelian. Finally, for $n \geq 3$, it is sufficient to note (again) that F_n is not a visible subgroup of F_2.

At the opposite extreme, a well-known result [9, Lemma 18] states that the maximum rank of a free-abelian subgroup of a f.g. PC-group $\langle\, \Gamma \,\rangle$ coincides with the maximum size of a complete subgraph in Γ. An immediate corollary is that every (finite) complete graph is explicit.

In the last years, embedability between PC-groups has been a matter of growing interest and research (see [7], [9] and [2]). In particular, new examples of explicit graphs are known, such as the square C_4 (proved by Kambites in [7]), or the path on four vertices P_4 (proved by Kim and Koberda in [9]).

To end with, we just remark that our characterization theorem (Theorem 2.1) immediately provides a new member of this family.

Corollary 2.2. *The path on three vertices* P_3 *is explicit.* $\square$

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