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A quantitative Runge's Theorem in Riemann surfaces

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Resum *(CAT)*
Donem una versió quantitativa del teorema de Runge per a superfícies de Riemann, la qual inclou una fita superior de l'ordre dels pols. Juguen un paper essencial tant la funció de Green com les estimacions L^2 ponderades per a l'equació de Cauchy-Riemann no homogènia.

Abstract *(ENG)*
We give a quantitative version of Runge's theorem for Riemann surfaces that includes an upper bound of the order of the poles. Green's Functions and the weighted L^2-estimates for the inhomogeneous Cauchy–Riemann equation play an essential role.

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1. Introduction

In complex analysis, Runge's theorem (also known as Runge's approximation theorem) is named after the German mathematician Carl Runge who first proved it in the year 1885. It states the following:

Theorem 1.1. *Let K be a compact subset of the extended complex plane $\hat{\mathbb{C}} \equiv \mathbb{C} \cup \{\infty\}$ and suppose that f is holomorphic on an open set containing K. Let Q be a subset of $\hat{\mathbb{C}} \backslash K$ such that each connected component of $\hat{\mathbb{C}} \backslash K$ contains a point of Q. Then f can be approximated uniformly on K by rational functions with poles in Q.*

In particular, if K is a compact subset of the complex plane, and if the complement of K is connected, then each holomorphic function in a neighborhood of K can be approximated uniformly on K by polynomials. Runge's theorem has many applications in the theory of functions of a complex variable and in functional analysis. The proofs of this theorem and its applications can be found in any monograph of complex analysis as [5, 10, 11].

The main contribution of this paper is a new proof of Runge's theorem. The proof generalizes to the following theorem for Riemann surfaces.

Theorem 1.2. *Let X be a compact Riemann surface and let $K \subset X$ be a compact subset. Moreover, let Q be any subset of $X \backslash K$ which contains precisely one point from each connected component of $X \backslash K$. Then any holomorphic function on a neighborhood of K can be approximated uniformly on K by meromorphic functions on X whose poles lie in Q.*

The proof is based on Hörmander's L^2-estimates for the inhomogeneous $\bar{\partial}$-equation. More precisely, a smooth approximant is obtained by multiplying f by a cut-off function χ adapted to K. Then, the rational function of the form $g = \chi f - u$ will provide us the desired approximation of f on K. This leads to the $\bar{\partial}$-equation $\bar{\partial} u = f \bar{\partial} \chi$, where u must be small on K and with controlled growth near Q (so that g is meromorphic with poles only on Q). This is achieved by constructing an appropriate subharmonic weight with singularities located on Q and applying Hörmander's result.

A clear advantage of this method is that it controls the order of the poles. This explains the word "quantitative" in the title.

In the next section we give a detailed account of the Riemann sphere case $\hat{\mathbb{C}}$ in order to illustrate the main difficulties. This case, although easier than the general case of Riemann surfaces, introduces the general procedure and is conceptually easier to understand. The general case is dealt with in Section 3.

2. A particular case: the Riemann sphere

Let K be a compact set in $\hat{\mathbb{C}}$ such that $\hat{\mathbb{C}} \backslash K$ has finitely many regions $\Omega_1, \dots, \Omega_n$. We fix one point z_i in each of the components Ω_i. In order to prove Runge's theorem we need to see that given $f \in \mathcal{H}(K)$ [1] and $\varepsilon > 0$ there is a rational function $g = p/q$ with poles only in $z_1, z_2, \dots, z_n$ such that $\sup_K |f - g| \leq \varepsilon$.

Remark. The degree of q depends on f, the position of the points $z_1, \dots, z_n$ and ε. However, the precise dependence is not clear when looking at the standard proofs. Our goal is to prove Runge's theorem with control on the poles of g.

[1] By definition, $f \in \mathcal{H}(K)$ if there exists an open set U with $K \subset U \subset \hat{\mathbb{C}}$ s.t. f is holomorphic on U.

The main tools in our proof are Hörmander L^2-estimates and potential theory, for which we refer to [3, 4, 9] respectively.

To make it simpler we will split the proof into several steps.

Step 1. Green's function for Ω_i with pole z_i. Recall the definition of Green's function.

Definition 2.1. Let D be a proper subdomain of $\hat{\mathbb{C}}$. A Green's function for D is a map $g_D : D \times D \longrightarrow (-\infty, \infty]$ such that for each $\omega \in D$:

(a) $g_D(\cdot, \omega)$ is harmonic on $D\backslash\{\omega\}$ and bounded outside any neighbourhood of ω.

(b) $g_D(\omega, \omega) = \infty$ and $\lim_{z\to\omega} g_D(z, \omega) = \begin{cases} \log|z| + O(1) & \omega = \infty \\ -\log|z-\omega| + O(1) & \omega \neq \infty \end{cases}$.

(c) $\lim\limits_{z\to\zeta} g_D(z, \omega) = 0$ for nearly everywhere [2] $\zeta \in \partial D$.

Specifically, we consider the case in which $D = \Omega_i$ and $\omega = z_i$. Since Ω_i is a regular[3] domain in $\hat{\mathbb{C}}$ such that $\partial\Omega_i$ is non-polar[4], there exists a unique Green's function

$$G_i(z) := g_{\Omega_i}(z, z_i) \quad z \in \Omega_i.$$

In particular,

- $G_i(\cdot) = g_{\Omega_i}(\cdot, z_i) > 0$,
- $\lim_{z\to\zeta} G_i(z) = \lim_{z\to\zeta} g_{\Omega_i}(z, z_i) = 0 \quad \text{for } \zeta \in \partial\Omega_i$.

Moreover, we can extend G_i to the whole $\hat{\mathbb{C}}$ by declaring $G_i \equiv 0$ outside of Ω_i, $i = 1, \dots, n$.

As $f \in \mathcal{H}(K)$ and $G_i(z) \in \mathcal{C}(\overline{\Omega}_i\backslash\{z_i\})$ with $G_i(z) \geq 0$ in $\overline{\Omega}_i$ and $G_i(z) \equiv 0$ on $\partial\overline{\Omega}_i$, there exists $\delta_i > 0$ small enough such that f is defined in $\widetilde{U}_i := \{z \in \Omega_i : G_i(z) < \delta_i\}$, see Figure 1.

Take now δ such that $0 < \delta \leq \min\{\delta_1, \dots, \delta_n\}$, so that f is defined in $U_i := \{z \in \Omega_i : G_i(z) < \delta\}$ for every $i = 1, \dots, n$. At the moment δ is freely chosen in the region $0 < \delta \leq \min\{\delta_1, \dots, \delta_n\}$. However, further on we will give δ a specific value.

So far we have $G_i(z) - \delta \in \mathcal{C}(\overline{\Omega}_i\backslash\{z_i\})$ with $G_i(z) - \delta \equiv 0$ in $\partial\overline{U}_i$. The next step is to extend $G_i(z) - \delta$ to the entire plane $\hat{\mathbb{C}}$; define finally $\mathcal{G}_i(\cdot) : \hat{\mathbb{C}} \longrightarrow [0, \infty]$ as:

$$\mathcal{G}_i(z) := \begin{cases} G_i(z) - \delta & \text{if } z \in \Omega_i\backslash\overline{U}_i, \\ 0 & \text{if } z \notin \Omega_i\backslash\overline{U}_i. \end{cases}$$

[2] A property is said to hold nearly everwhere (n.e.) on a subset S of $\mathbb{C}$ if it holds everywhere on $S\backslash E$, for some Borel polar set E.

[3] Let D be a proper subdomain of $\hat{\mathbb{C}}$, and let $\xi_0 \in \partial D$. A barrier at ξ_0 is a subharmonic function b defined on $D \cap N$, where N is an open neighborhood of ξ_0 satisfying

$$b < 0 \text{ on } D \cap N \quad \text{and} \quad \lim_{z\to\xi_0} b(z) = 0$$

A boundary point at which a barrier exists is called regular. If every $\xi \in \partial D$ is regular, then D is called a regular domain.

[4] In the area of classical potential theory, polar sets are the "negligible sets", similar to the way in which sets of measure zero are the negligible sets in measure theory.

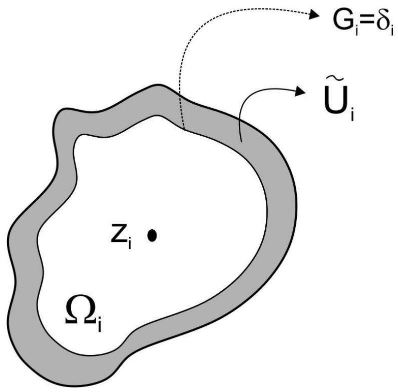

Figure 1: Definition of $\widetilde{U}_i$.

Step 2. Construction of a subharmonic weight $\phi_\delta(z)$ in $\hat{\mathbb{C}}\backslash\{z_1, \dots, z_n\}$. Notice that $\mathcal{G}_i(\cdot) \equiv G_i(\cdot) - \delta$ on $\Omega_i\backslash\overline{U}_i$. In particular $\mathcal{G}_i$ is subharmonic in $(\Omega_i\backslash\overline{U}_i)\backslash\{z_i\}$. We conclude that $\mathcal{G}_i(\cdot)$ is subharmonic on $\hat{\mathbb{C}}\backslash\{z_1, \dots, z_n\}$, $i = 1, \dots, n$.

We see that $\phi_\delta(z) := \max\{0, \mathcal{G}_1(z), \mathcal{G}_2(z), \dots, \mathcal{G}_n(z)\}$ is subharmonic on $\hat{\mathbb{C}}\backslash\{z_1, \dots, z_n\}$, since the maximum of subharmonic functions is again subharmonic. The explicit expression of ϕ_δ is

$$\phi_\delta(z) = \begin{cases} \mathcal{G}_i(z) = G_i(z) - \delta & \text{if } z \in \Omega_i\backslash\overline{U}_i \\ 0 & \text{if } z \in K \cup \bigcup_{j=1}^N \overline{U}_j. \end{cases}$$

Step 3. A cut-off function adapted to K. In this step we construct a suitable smooth cut-off function χ, so that χf is a smooth extension of f which is still holomorphic in K.

Consider a parameter $t > 0$ small enough, which will be fixed later. We look for a smooth cut-off function $\chi(z) \in \mathcal{C}^\infty(\mathbb{C})$ such that $\chi = 0$ when $G_i(z) \geq \delta$ and $\chi = 1$ when $z \in K$ or $G_i(z) \leq \delta - t\delta/2$, see Figure 2. The easier way to achieve this is to take $\chi(z) = \varphi_\delta(\sum_i G_i(z))$ with $\varphi_\delta \in \mathcal{C}^\infty(\mathbb{R})$ such that:

$$\varphi_\delta(x) = \begin{cases} 1 & \text{if} \quad x \leq \delta - t\delta/2, \\ 0 & \text{if} \quad x \geq \delta. \end{cases}$$

Since by construction

$$\sum_i G_i(z) = \begin{cases} G_i(z) & \text{if } z \in \Omega_i, \\ 0 & \text{otherwise,} \end{cases}$$

we obtain:

$$\chi(z) = \begin{cases} 0 & \text{if} \quad z \in \{z \in \Omega_i : G_i(z) \geq \delta\}, \\ 1 & \text{if} \quad z \in K \cup \bigcup_{i=1}^n \{z \in \Omega_i : G_i(z) \leq \delta - t\delta/2\}. \end{cases}$$

Note that

$$\mathrm{Supp}(\bar{\partial}\chi) \subset \bigcup_{i=1}^n \{z \in \Omega_i : \delta - t\delta/2 \leq G_i(z) \leq \delta\}$$

and

$$|\bar{\partial}\chi| \sim |1/(t\delta/2)|. \tag{1}$$

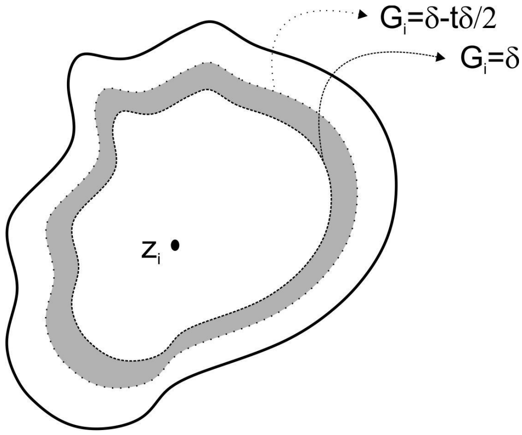

Figure 2: Definition of χ.

Step 4. The $\bar{\partial}$-equation. Here the smooth extension χf will be corrected, with the help of an appropriate solution to a $\bar{\partial}$-equation, to make the resulting function rational with poles on z_j. The function approximating f will be of the form $g = \chi f - u$, where $\bar{\partial} u = f \bar{\partial}\chi$ on $\hat{\mathbb{C}}\backslash\{z_1, \ldots, z_n\}$.

We will use the solution u given by the following theorem of Hörmander's. This solution has minimal norm in $L^2(e^{-\phi})$.

Theorem 2.2. [3, pp. 13] *Let Ω be a domain in $\hat{\mathbb{C}}$ and suppose $\phi \in \mathbb{C}^2(\Omega)$ with $\Delta\phi \geq 0$. Then, for any $f \in L^2_{\text{loc}}(\Omega)$ there is a solution u to $\bar{\partial} u = f$ satisfying*

$$\int |u|^2 e^{-\phi} \leq \int \frac{|f|^2}{\Delta\phi} e^{-\phi}.$$

Step 5. The measure $\Delta\phi_\delta \geq 0$ is the harmonic measure of $\{G_i = \delta\}$ with respect to z_i in $\Omega_i \backslash \overline{U}_i$. Hörmander's theorem provides a solution u with

$$\int |u|^2 e^{-\phi} \leq \int \frac{1}{\Delta\phi} |f \bar{\partial}\chi|^2 e^{-\phi},$$

whenever $\Delta\phi \geq 0$.

Our first candidate is $\phi = \phi_\delta$, which is subharmonic on $\hat{\mathbb{C}}\backslash\{z_1, \ldots, z_n\}$. We see that the Radon measure $\Delta\phi_\delta$ has

$$\Delta\phi_\delta\big|_{K\cup U_1\cup\cdots\cup U_n} \equiv 0 \quad \text{and} \quad \Delta\phi_\delta\big|_{(\Omega_i\backslash\overline{U}_i)\backslash\{z_i\}} \equiv 0 \qquad i = 1, 2, \ldots, n.$$

Therefore, $\Delta\phi_\delta$ is supported in the union $\bigcup_{i=1}^n \partial U_i$, that is, in the curve $\{G_i(z) = \delta\}$.

We shall see next that $\Delta\phi_\delta$ is the harmonic measure of $\{G_i = \delta\}$ with respect to z_i in the domain $\Omega_i \backslash \overline{U}_i$.

Definition 2.3. Let D be a proper subdomain of $\hat{\mathbb{C}}$, and denote by $\mathcal{B}(\partial D)$ the σ-algebra of Borel subsets of ∂D. A harmonic measure for D is a function $\omega_D : D \times \mathcal{B}(\partial D) \longrightarrow [0, 1]$ such that:

(a) for each $z \in D$, the map $B \mapsto \omega_D(z, B)$ is a Borel measure on ∂D,

(b) if $\phi : \partial D \longrightarrow \mathbb{R}$ is a continuous function, then $H_D\phi = P_D\phi$ on D, where $P_D\phi$ is the generalized Poisson integral and $H_D\phi$ is the Perron function of ϕ on D.

The harmonic measure of $E \in \mathcal{B}(\partial D)$ at $z \in D$ relative to D is the Perron solution $u(z)$ of the Dirichlet problem in D with boundary values 1 on E and 0 on $\partial D \backslash E$. Summarising, if χ_E denotes the indicator function of $E \subset \partial D \backslash E$ then

$$u(z) = \sup\Big\{ v(z) : v \text{ subharmonic in } D \text{ and } \limsup_{\omega \to \zeta} v(\omega) < \chi_E(\zeta) \text{ for } \zeta \in \partial D \Big\}.$$

We shall prove now that

$$\Delta \phi_\delta(\zeta) = 2\pi \sum_{i=1}^{n} \omega_{\Omega_i \backslash \overline{U}_i}(z_i, \zeta), \tag{2}$$

where $\omega_{\Omega_i \backslash \overline{U}_i} : \Omega_i \backslash \overline{U}_i \times \mathcal{B}\left(\partial(\Omega_i \backslash \overline{U}_i)\right) \longrightarrow [0,1]$ is the harmonic measure for $\Omega_i \backslash \overline{U}_i$, and we denote by $\mathcal{B}(\partial(\Omega_i \backslash \overline{U}_i))$ the σ-algebra of Borel subsets of $\partial(\Omega_i \backslash \overline{U}_i)$.

Take in the definition $D = \Omega_i \backslash \overline{U}_i$ and fix $z_i \in \Omega_i \backslash \overline{U}_i$. Since $\partial\left(\Omega_i \backslash \overline{U}_i\right)$ is non-polar there exists a unique harmonic measure satisfying (a) and (b). We then repeat the same reasoning for each of the "holes".

By definition, the generalized Laplacian acts on test functions as:

$$\int_D \psi \Delta \phi_\delta = \int_D \phi_\delta \Delta \psi \, dA \qquad \psi \in C_c^\infty(D) \qquad D = \hat{\mathbb{C}} \backslash \{z_1, \dots, z_n\}.$$

Since $\sup(\Delta \phi_\delta) \subset \bigcup_{i=1}^n \{z \in \Omega_i : G_i(z) = \delta\}$ and $\phi_\delta \equiv 0$ in $K \cup \overline{U}_1 \cup \cdots \cup \overline{U}_n$, if $\psi \in C_c^\infty(D)$, the previous expression becomes

$$\sum_{i=1}^{n} \int_{\{z \in \Omega_i : G_i(z) = \delta\} \equiv \partial(\Omega_i \backslash \overline{U}_i)} \psi \Delta \phi_\delta = \sum_{i=1}^{n} \int_{(\Omega_i \backslash \overline{U}_i) \backslash \{z_i\}} \phi_\delta \Delta \psi \, dA.$$

Thus, in order to prove (2) we need to see that

$$\int_{(\Omega_i \backslash \overline{U}_i) \backslash \{z_i\}} \phi_\delta(\zeta) \Delta \psi(\zeta) \, dA = 2\pi \int_{\{z \in \Omega_i : G_i(z) = \delta\} \equiv \partial(\Omega_i \backslash \overline{U}_i)} \psi(\zeta) \, d_{\omega_{\Omega_i \backslash \overline{U}_i}}(z_i, \zeta).$$

To see this, we are going to use the relationship between the harmonic measure and the normal derivative of Green's function, which is the Poisson kernel.

Theorem 2.4. [5, pp. 409] *Let D be a bounded domain with piecewise smooth boundary. Fix $\zeta \in D$ and let $g_D(z, \zeta)$ be the Green's function for D (with pole at ζ). Then for any Borel mesurable set $B \in \mathcal{B}(\partial D)$ we have*

$$-\frac{1}{2\pi} \int_B \frac{\partial g_D}{\partial n}(z, \zeta) \, ds = \omega_D(\zeta, B).$$

Going back to the proof of the identity above, we consider the particular case $D = \Omega_i \backslash \overline{U}_i$ with $\zeta = z_i$, and repeating the computations for each hole, we get

$$-\frac{1}{2\pi} \int_B \frac{\partial \phi_\delta}{\partial n}(z, z_i) \, ds = \omega_{\Omega_i \backslash \overline{U}_i}(z_i, B),$$

for $i = 1, \dots, n$. Therefore, applying Stoke's theorem, we obtain

$$\begin{aligned} 2\pi \int_{\partial(\Omega_i \setminus \overline{U}_i)} \psi(\zeta)\, d_{\omega_{\Omega_i \setminus \overline{U}_i}}(z_i, \zeta) &= 2\pi \left[-\frac{1}{2\pi} \int_{\partial(\Omega_i \setminus \overline{U}_i)} \psi(\zeta) \frac{\partial \phi_\delta}{\partial n}(z, z_i)\, ds \right] \\ &= 2\pi \left[\psi(z_i) + \frac{1}{2\pi} \int_{\Omega_i \setminus \overline{U}_i} \phi_\delta(\zeta) \Delta\psi(\zeta)\, dA \right] \\ &= 2\pi\psi(z_i) + \int_{\Omega_i \setminus \overline{U}_i} \phi_\delta(\zeta)\Delta\psi(\zeta)\, dA. \end{aligned}$$

Taking $\varepsilon > 0$ small enough and splitting the last integral

$$\begin{aligned} \int_{\Omega_i \setminus \overline{U}_i} \phi_\delta(\zeta)\Delta\psi(\zeta)\, dA &= \int_{(\Omega_i \setminus \overline{U}_i)\setminus B(z_i,\varepsilon)} \phi_\delta(\zeta)\Delta\psi(\zeta)\, dA + \int_{B(z_i,\varepsilon)} \phi_\delta(\zeta)\Delta\psi(\zeta)\, dA \\ &= \lim_{\varepsilon\to 0^+} \left[\int_{(\Omega_i \setminus \overline{U}_i)\setminus B(z_i,\varepsilon)} \phi_\delta(\zeta)\Delta\psi(\zeta)\, dA + \int_{B(z_i,\varepsilon)} \phi_\delta(\zeta)\Delta\psi(\zeta)\, dA \right] \\ &= \int_{(\Omega_i \setminus \overline{U}_i)\setminus \{z_i\}} \phi_\delta(\zeta)\Delta\psi(\zeta)\, dA + \lim_{\varepsilon\to 0^+} \int_{B(z_i,\varepsilon)} \phi_\delta(\zeta)\Delta\psi(\zeta)\, dA. \end{aligned}$$

We compute the second integral:

$$\begin{aligned} \int_{B(z_i,\varepsilon)} \phi_\delta(\zeta)\Delta\psi(\zeta)\, dA &= \int_{B(z_i,\varepsilon)} \left(\phi_\delta(\zeta) - \log\frac{1}{|\zeta - z_i|} + \log\frac{1}{|\zeta - z_i|} \right) \Delta\psi(\zeta)\, dA \\ &= \int_{B(z_i,\varepsilon)} \left(\phi_\delta(\zeta) - \log\frac{1}{|\zeta - z_i|} \right) \Delta\psi(\zeta)\, dA - \int_{B(z_i,\varepsilon)} \log|\zeta - z_i|\, \Delta\psi(\zeta)\, dA \\ &= \int_{B(z_i,\varepsilon)} \underbrace{\Delta\left(\phi_\delta(\zeta) - \log\frac{1}{|\zeta - z_i|} \right)}_{0} \psi(\zeta)\, dA - \int_{B(z_i,\varepsilon)} \psi(\zeta)\, \underbrace{\Delta(\log|\zeta - z_i|)}_{2\pi\delta_{z_i}(\zeta)} \\ &= -2\pi\psi(z_i), \end{aligned}$$

since $\phi_\delta(\zeta)$ has a logarithmic singularity and so $\phi_\delta(\zeta) - \log\frac{1}{|\zeta - z_i|}$ is harmonic in $\{z_i\}$, and also on $B(z_i,\varepsilon)$ for ε small enough. Thus, we have

$$2\pi \int_{\partial(\Omega_i \setminus \overline{U}_i)} \psi(\zeta)\, d_{\omega_{\Omega_i \setminus \overline{U}_i}}(z_i, \zeta) = \int_{(\Omega_i \setminus \overline{U}_i)\setminus\{z_i\}} \phi_\delta(\zeta)\Delta\psi(\zeta)\, dA,$$

as desired.

Step 6. The curve $\{G_i = \delta\}$ is smooth and $\Delta\phi_\delta$ is comparable to the length of the curve. We have seen that $\Delta\phi_\delta$ is the harmonic measure of $\{G_i = \delta\}$ with respect to z_i in the domain $\Omega_i \setminus \overline{U}_i$. Here we prove that the curve $\{G_i = \delta\}$ is smooth; later we shall use the smoothness to prove that $\Delta\phi_\delta$ is comparable to the length of the curve.

Recall that $F \equiv \sum_{i=1}^n G_i$ is harmonic in its domain of definition $\bigsqcup_{i=1}^n (\Omega_i \setminus \overline{U}_i)\setminus\{z_i\}$. Let us fix here δ, taking $0 < \delta \le \min\{\delta_1, \dots, \delta_n\}$ such that if $F(z) = \delta$, then the vector $\nabla F(z) = (F_x(z), F_y(z)) \ne (0,0)$. To prove that such δ exists we recall Sard's theorem.

Theorem 2.5. [1, pp. 34] *Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be $\mathcal{C}^k$ with $k \ge 2$. Let X denote the critical set of f, $X = \{x \in \mathbb{R}^2 : \nabla f(x) = (0,0)\}$. Then the image $f(X)$ has Lebesgue measure 0 in $\mathbb{R}$.*

The existence of δ is proved by contradiction. Assume that for all $\delta \in (0, \min\{\delta_1, \dots, \delta_n\}]$ there exists at least a point z_δ in the curve $\{F = \delta\}$ such that $\nabla F(z_\delta) = (0,0)$. Define $M := \{z_\delta : 0 <$

$\delta \leq \min\{\delta_1, \dots, \delta_n\}\}$. By Sard's theorem $F(M)$ has measure zero; however it is clear that $F(M) = (0, \min\{\delta_1, \dots, \delta_n\}]$, so $m(F(M)) = \min\{\delta_1, \dots, \delta_n\} > 0$, and we get a contradiction.

Therefore, according to the implicit function theorem, the curve $\{F = \delta\}$ (or equivalently $\{G_i = \delta\}$) is smooth.

In order to prove that $\Delta\phi_\delta$ is comparable to the length of the curve we use that the gradient at a point is perpendicular to the level set at that point:

$$\nabla\phi_\delta \equiv -\frac{\partial\phi_\delta}{\partial n}.$$

Since $\nabla\phi_\delta \equiv \nabla F \neq (0,0)$ over the smooth curve $\{G_i = \delta\}$, we can extend this property by continuity to a "thin strip" $\{z : \delta - k\delta \leq G_i(z) \leq \delta\}$. Using that $0 \neq -\frac{\partial\phi_\delta}{\partial n}(z) \equiv \nabla\phi_\delta(z)$ on the "thin strip", we see that there exist $c_1 < c_2 < 0$ such that $c_1 \leq \frac{\partial\phi_\delta}{\partial n} \leq c_2 < 0$ on it. Hence:

$$-\frac{c_1}{2\pi} l(B) = -\frac{1}{2\pi}\int_B c_1\, ds \geq -\frac{1}{2\pi}\int_B \frac{\partial\phi_\delta}{\partial n}(z,\zeta)\, ds = \omega_D(\zeta, B),$$
$$-\frac{c_2}{2\pi} l(B) = -\frac{1}{2\pi}\int_B c_2\, ds \leq -\frac{1}{2\pi}\int_B \frac{\partial\phi_\delta}{\partial n}(z,\zeta)\, ds = \omega_D(\zeta, B).$$

Combining both inequalities, we obtain

$$-c_2/2\pi\, l(B) = C_2 \cdot l(B) \leq \omega_D(\zeta, B) \leq C_1 \cdot l(B) = -c_1/2\pi\, l(B).$$

As a result, the harmonic measure is comparable to the length of the curve and consequently, $\Delta\phi_\delta$ is also comparable to the length of the curve.

Remark. In order to derive an estimate from Hörmander's theorem, it is necessary to have $\Delta\phi_\delta \geq \lambda > 0$, for some $\lambda \in \mathbb{R}$.

Step 7. A function ϕ as an average of ϕ_δ's and a lower bound for $\Delta\phi$. The most natural approach is to replace the weight ϕ_δ by an average of ϕ_δ which distributes the Laplacian from the border to a neighborhood. To do this, we consider

$$\phi := \frac{1}{t\delta}\int_{\delta - t\delta}^{\delta} \phi_s\, ds.$$

Splitting the domain of definition into three regions we see that

$$\phi(z) = \begin{cases} 0 & \text{in } K \cup \{z \in \Omega_i : G_i(z) < \delta - t\delta\}, \\ \frac{1}{2t\delta}[G_i(z) - (\delta - t\delta)]^2 & \text{in } \{z \in \Omega_i : \delta - t\delta \leq G_i(z) \leq \delta\}, \\ G_i(z) - (\delta - t\delta/2) & \text{in } \{z \in \Omega_i : G_i(z) > \delta\}. \end{cases}$$

In particular ϕ is harmonic in $K \cup \{z \in \Omega_i : G_i(z) < \delta - t\delta$ or $G_i(z) > \delta\}$ and the support of $\Delta\phi$ is contained on the "thin strips" $\{z \in \Omega_i : \delta - t\delta \leq G_i(z) \leq \delta\}$.

To prove that $\Delta\phi$ is bounded below in $\{z \in \Omega_i : \delta - t\delta \leq G_i(z) \leq \delta\}$, we use the expression $\phi(z) = \frac{1}{2t\delta}[G_i(z) - (\delta - t\delta)]^2$ in $\{z \in \Omega_i : \delta - t\delta \leq G_i(z) \leq \delta\}$. We have

$$\Delta\phi(z) = \frac{1}{2t\delta}\left[\Delta(G_i(z)^2) + \overbrace{\Delta(\delta - t\delta)^2}^{0} - 2(\delta - t\delta)\overbrace{\Delta G_i(z)}^{0}\right] = \frac{1}{2t\delta}\Delta(G_i(z)^2)$$
$$= \frac{1}{t\delta}\left[\left(\frac{\partial G_i}{\partial x}\right)^2 + \left(\frac{\partial G_i}{\partial y}\right)^2\right](z).$$

The last identity follows from the fact that $\Delta(G_i)^2 = 2(\nabla G_i)^2$, which is a well-known property of harmonic functions:

$$\Delta(G_i)^2 = \nabla(\nabla G_i^2) = \nabla(2G_i\nabla G_i) = 2(\nabla G_i\nabla G_i + G_i\Delta G_i) = 2(\nabla G_i)^2.$$

However, $\nabla G_i(z) \neq 0$ in $\{z \in \Omega_i : \delta - t\delta \leq G_i(z) \leq \delta\}$, so $\Delta(G_i(z)^2) > 0$ too. Since G_i is continuous on it, according to Weierstrass' theorem there exists $\widetilde{\lambda} > 0$ such that $\Delta(G_i(z)^2) \geq \widetilde{\lambda}$. Therefore, there exists $\lambda > 0$ such that $\Delta\phi \geq \lambda$ on $\{z \in \Omega_i : \delta - t\delta \leq G_i(z) \leq \delta\}$ and in particular on $\operatorname{Supp}\bar{\partial}\chi$, since we have the following inclusion of sets

$$\operatorname{Supp}\bar{\partial}\chi \subset \{z \in \Omega_i : \delta - t\delta/2 \leq G_i(z) \leq \delta\} \subset \{z \in \Omega_i : \delta - t\delta \leq G_i(z) \leq \delta\}.$$

Furthermore,

$$\Delta\phi = \frac{1}{t\delta}\int_{\delta-t\delta}^{\delta} \Delta\phi_s \, ds.$$

Since, as we have seen, $\Delta\phi_\delta$ is the harmonic measure, which is comparable with the length of the curve, we obtain

$$\Delta\phi(E) = \frac{1}{t\delta}\int_{\delta-t\delta}^{\delta} \Delta\phi_s(E)\, ds \approx m(E \cap \text{“strip”}).$$

Step 8. Hörmander's estimate with the weight ϕ and the final approximant function. Since ϕ is subharmonic on $\hat{\mathbb{C}}\backslash\{z_1,\dots,z_n\}$ and satisfies $\Delta\phi \geq \lambda > 0$ on $\operatorname{Supp}\bar{\partial}\chi$, Hörmander's theorem with the subharmonic weight function $M\phi$, with $M >> 0$ to be fixed later, yields:

$$\begin{aligned}\int_{\hat{\mathbb{C}}\backslash\{z_1,\dots,z_n\}} |u|^2 e^{-M\phi} &\leq \int_{\hat{\mathbb{C}}\backslash\{z_1,\dots,z_n\}} |f\bar{\partial}\chi|^2 \frac{e^{-M\phi}}{M\Delta\phi} = \int_{\operatorname{Supp}\bar{\partial}\chi} |f\bar{\partial}\chi|^2 \frac{e^{-M\phi}}{M\Delta\phi}\\ &\leq \frac{1}{M\lambda}\int_{\cup_i\{z\in\Omega_i:\delta-t\delta/2\leq G_i(z)\leq\delta\}} |f\bar{\partial}\chi|^2 e^{-M\phi}\\ &\leq \frac{1}{M\lambda}\int_{\cup_i\{z\in\Omega_i:\delta-t\delta/2\leq G_i(z)\leq\delta\}} |f\bar{\partial}\chi|^2 e^{-Mt\delta/8}.\end{aligned}$$

The last inequality is a consequence of the fact that if $z \in \Omega_i$ with $\delta - t\delta/2 \leq G_i(z) \leq \delta$, then $\phi(z) \geq t\delta/8$.

Taking $M = \frac{16}{t\delta}\log\left(1/(\pi r^2)^{1/2}\varepsilon\right)$, where r will be fixed in the next lines, we see that $e^{-Mt\delta/8} = \pi r^2 \cdot \varepsilon^2$. This and (1) show that

$$\int_{\hat{\mathbb{C}}\backslash\{z_1,\dots,z_n\}} |u|^2 e^{-M\phi} \leq \left(\frac{4}{M\lambda(t\delta)^2}\int_{\cup_i\{z\in\Omega_i:\delta-t\delta/2\leq G_i(z)\leq\delta\}} |f|^2\right)\pi r^2\,\varepsilon^2 \lesssim \pi r^2\,\varepsilon^2.$$

The last inequality follows from the fact that f is holomorphic in the domain of integration, which is compact. Using $\phi|_{K\cup U_i} \equiv 0$ we have that

$$\int_{K\cup U_i} |u|^2 \leq \int_{\hat{\mathbb{C}}\backslash\{z_1,\dots,z_n\}} |u|^2 e^{-M\phi},$$

and so

$$\int_{K\cup U_i} |u|^2 \lesssim \pi r^2\,\varepsilon^2.$$

We wish to take advantage of the last inequality to deduce that $\sup_K |u| \lesssim \varepsilon$. Since $\bar{\partial}u = f\bar{\partial}\chi$ on $\hat{\mathbb{C}}\backslash\{z_1,\dots,z_n\}$ and $\chi \equiv 1$ in a small neighborhood of K, we have that $\bar{\partial}\chi = 0$ there. It follows that

$\bar{\partial} u = 0$ and thus u is holomorphic on this small neighborhood. In particular $|u|^2$ is subharmonic and, by the sub-mean value property, for $r > 0$ small enough (say, $0 < r < d(K, U_i^c)$, $i = 1, \dots, n$), we have

$$|u(z)|^2 \leq \frac{1}{\pi r^2} \int_{B(z,r)} |u(\zeta)|^2 \, dA(\zeta) \leq \frac{1}{\pi r^2} \int_{K \cup U_i} |u(\zeta)|^2 \, dA(\zeta) \qquad (z \in K).$$

Therefore,

$$\sup_K |u|^2 \lesssim \pi r^2 \varepsilon^2 / \pi r^2 = \varepsilon^2.$$

Using the previous inequality and the fact that $\chi \equiv 1$ in K, we get

$$\sup_K |u| = \sup_K |g - \chi f| = \sup_K |g - f| \lesssim \varepsilon.$$

Moreover, as $g = \chi f - u$, we get $\bar{\partial} g = \chi \bar{\partial} f$. Since $f \in \mathcal{H}(K \cup U_i)$ we have $\bar{\partial} f \equiv 0$ in $K \cup U_i$. This and the fact that $\chi \equiv 0$ in $\Omega \backslash U_i$ show that $\chi \bar{\partial} f \equiv 0$. Therefore $\bar{\partial} g = 0$ in $\hat{\mathbb{C}} \backslash \{z_1, \dots, z_n\}$, or equivalently g is holomorphic in $\hat{\mathbb{C}} \backslash \{z_1, \dots, z_n\}$.

Let us see next that the singularities z_i must be poles, and let's quantify their order.

Consider $B_i \equiv B(z_i, r_i) \backslash \{z_i\}$. It is clear that $\chi = 0$ in B_i (for r_i small enough). Near any of the singularities z_i, $g = u$, and thus $\int_{B_i} |g|^2 e^{-M\phi} < \infty$. On the other hand, the sizes of ϕ and ϕ_δ are very similar in size. In B_i

$$\phi_\delta(z) \approx \begin{cases} -\log|z - z_i| + O(1) & z_i \neq \infty, \\ \log|z| + O(1) & z_i = \infty. \end{cases}$$

As a consequence,

$$e^{-M\phi} \approx e^{-M\phi_\delta} \approx \begin{cases} e^{M \log|z - z_i|} = |z - z_i|^M & z_i \neq \infty \\ e^{-M \log|z|} = |z|^{-M} & z_i = \infty \end{cases}$$

and therefore,

$$\int_{B_i} |g|^2 e^{-M\phi} \approx \begin{cases} \int_{B_i} |g|^2 |z - z_i|^M < \infty & z_i \neq \infty \Longrightarrow |g(z)| \approx |z - z_i|^{-\alpha/2} \text{ with } \alpha \leq M \\ \int_{B_i} |g|^2 |z|^{-M} < \infty & z_i = \infty \Longrightarrow |g(z)| \approx |z|^{\alpha/2} \text{ with } \alpha \leq M. \end{cases}$$

Hence, g can only have poles on the z_i and on ∞, and the order of such poles is at most $M/2$.

Finally, in the Riemann sphere $\hat{\mathbb{C}}$ the field of meromorphic functions is simply the field of rational functions over the complex field. Therefore, the meromorphic function g can be written as the rational function p/q, where the degree of g is smaller than $M/2$ with $M \approx \log(1/\varepsilon)$. Thus, the degree of q is smaller than $C \log(1/\varepsilon)$ with

$$C = \tfrac{8}{t\delta} \tfrac{\log\left((\pi r^2)^{1/2} \varepsilon\right)}{\log \varepsilon} \approx \tfrac{8}{t\delta}.$$

We finish with the explanation of the geometric meaning of the parameter δ, which gives the size of the final estimate. By hypothesis, $f \in \mathcal{H}(K)$, which means that there exists U open with $K \subset U \subset \hat{\mathbb{C}}$ and such that f is holomorphic in U. Then δ measures the size of this extension, i.e

$$\delta \approx d(K, U^c).$$

As expected, the estimate shows that the order of the poles is inversely proportional to the size of the extension. A greater holomorphy domain for a function will result in a smaller order of its poles. In the extremal case where the starting function is entire, the order of the poles vanishes and the approximating function is polynomial.

3. The general case: Riemann surfaces

In the previous section, we have proved the classical Runge's theorem. This proof, although conceived for compact subsets of the Riemann sphere has the advantage that works *mutatis mutandis* on any Riemann surface X, with meromorphic functions on X playing the role of rational functions.

The result we now present is essentially equivalent to the Behnke-Stein generalization of the Runge approximation theorem [2]. Let us highlight that the former is a key tool used in a wide amount of problems in the context of open Riemann surfaces. It can be stated in several ways, for instance:

Theorem 3.1. (Behnke-Stein [1948]) *Let X be a Riemann surface and K a compact subset of X. Every holomorphic function in a neighborhood of K is uniformly approximable on K by holomorphic functions on X if and only if $X \backslash K$ has no connected components with compact closure in X.*

If X is a compact Riemann surface, then the theorem is vacuous. This is the reason why we say "the Behnke-Stein theorem for open Riemann surfaces".

The Behnke-Stein theorem gives a relationship between analytical and topological results. We start assuming the analytical part and we will prove the topological implication. We do this by contradiction.

Assuming the approximation condition, we consider U a relatively compact component of the complement of K and fix p a point in U. Let us now use the following results:

Proposition 3.2. [8, pp. 31] *Let X and Y be Riemann surfaces. If $\Psi : X \longrightarrow Y$ is a nonconstant holomorphic mapping, then the fiber $\Psi^{-1}(x)$ over each point $x \in Y$ is discrete in X (i.e. $\Psi^{-1}(x)$ has no limit points in X).*

Proposition 3.3. [8, pp. 89] *Let P be a discrete subset of an open Riemann surface X. If $\xi_p \in \mathbb{C}$ for each $p \in P$, then there exists a function $h \in \mathcal{H}(X)$ with $h(p) = \xi_p$ for every $p \in P$.*

Let f be a holomorphic function that vanishes at p, let $P := f^{-1}(0)$ be the (discrete) zero set of f, and consider the holomorphic function h given by the previous proposition such that

$$h(P - \{p\}) = 0 \qquad \text{and} \qquad h(p) = 1.$$

Define the meromorphic function

$$g := h/f.$$

It is clear that g has only one singularity (the point p), so it is holomorphic on K. Furthermore, the previous assumptions assert that we can find a sequence $\{g_n\}_n$ of holomorphic functions on X converging uniformly on K to g: for all $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon)$ such that:

$$\sup_K |g - g_n| \le \varepsilon \qquad \forall n \ge n_0.$$

Then the sequence fg_n converges to h uniformly on K. Note that this result still holds for the boundary of U. Hence, by the maximum principle, if converges uniformly to h on U, that is

$$\sup_{\overline{U}} |h - fg_n| \le \tilde{\varepsilon} \qquad \forall n \ge n_0.$$

This contradicts the fact that fg_n vanishes at p. Therefore, U cannot be relatively compact.

Also, it is easy to see that the analytical part of Behnke-Stein follows from the topological part applying the following result [12], which is similar to classical Runge's theorem.

Theorem 3.4. *Let X be a compact Riemann surface, and $K \subset X$ a compact subset. Let Q be any subset of $X \backslash K$ which contains (precisely) one point q_i from each connected component W_i of $X \backslash K$. Then, any holomorphic function on a neighborhood of K can be approximated uniformly on K by meromorphic functions on X whose poles lie in Q.*

Remark. Any relatively compact open subset of *any* Riemann surface can also be regarded as an open subset of a compact Riemann surface.

Let us now state a couple of propositions that will be useful in our proof. First, recall that an open set $Y \subset X$ is said to be *geometrically Runge* in X if $X \backslash Y$ has no compact connected components. Moreover, we have the following exhaustion result.

Proposition 3.5. [14, pp. 127] *Suppose X is an open Riemann surface. Then there exists a sequence $Y_0 \subset\subset Y_1 \subset\subset Y_2 \subset\subset \cdots$ of relatively Runge domains with $\bigcup Y_\nu = X$ so that every Y_ν has a regular boundary.*

On the other hand, we recall here the so-called *Schottky double*. This construction can be performed as follows.

If M is a complex manifold with $C_1, C_2, \dots, C_m$ boundary components, one can consider an exact duplicate of it, say $\widetilde{M}$, with the same number of boundary components, say $\widetilde{C_1}, \widetilde{C_2}, \dots, \widetilde{C_m}$. Obviously, for each point $x \in M$ there is a "symmetric" point $\widetilde{x} \in \widetilde{M}$. The *Schottky double* M^* is formed as a disjoint union $M \sqcup \widetilde{M}$ and identifying each point $x \in C_i$ with its symmetric point $\widetilde{x} \in \widetilde{C_i}$ for $1 \leq i \leq m$.

Proposition 3.6. [13, pp. 217] *Let Y be relatively Runge domain with regular boundary on a Riemann surface X. Then the Schottky double Y^* obtained by gluing Y and its mirror image $\widetilde{Y}$ together along the boundary is a compact Riemann surface.*

We are now ready to prove the analytical implication. By hypotesis, X is an open Riemann surface and K a compact subset of X such that $X \backslash K$ has no connected components with compact closure in X. Denote by $\{Y_\nu\}_\nu$ the exhaustion sequence of X given by Proposition 3.5. It is clear that there exists ν_0 such that $K \subset Y_\nu, \widetilde{K} \subset \widetilde{Y_\nu}$ for all $\nu \geq \nu_0$.

Let Q be any subset of $\widetilde{X} \backslash \widetilde{K}$ which contains (precisely) one point $\widetilde{q_i}$ from each connected component $\widetilde{W_i}$ of $\widetilde{X} \backslash \widetilde{K}$. Considering Y_ν^* with $\nu \geq \nu_0$, we are in the hypothesis of theorem 3.4, therefore any holomorphic function on a neighborhood of K can be approximated uniformly on K by meromorphic functions on Y_ν^* whose poles lie in Q for all $\nu \geq \nu_0$.

In particular, for $\varepsilon > 0$ and $f \in \mathcal{H}(K)$ there exist $g_0 \in \mathcal{M}(Y_{\nu_0+1}^*)$ (so $g_0 \in \mathcal{H}(Y_{\nu_0+1})$) such that

$$\sup_{z \in K} |(f - g_0)(z)| < \varepsilon/2.$$

The idea is to apply the theorem to each pair $\{(Y_{\nu_0+(i+2)}, \overline{Y_{\nu_0+i}})\}_{i \geq 0}$ to see that there exist functions

$$\begin{aligned}
&g_1 \in \mathcal{H}(Y_{\nu_0+2}) && \text{such that} \quad |g_1 - g_0| < \varepsilon/2^2 \text{ on } \overline{Y_{\nu_0}},\\
&g_2 \in \mathcal{H}(Y_{\nu_0+3}) && \text{such that} \quad |g_2 - g_1| < \varepsilon/2^3 \text{ on } \overline{Y_{\nu_0+1}},\\
&\quad \dots && \quad \dots\\
&g_n \in \mathcal{H}(Y_{\nu_0+(n+1)}) && \text{such that} \quad |g_n - g_{n-1}| < \varepsilon/2^n \text{ on } \overline{Y_{\nu_0+(n-1)}}.
\end{aligned}$$

To check that the family $\mathcal{F} = \{g_n\}_{n \geq 0}$ is normal in X it is enough to prove that it is uniformly bounded in compact subsets of X. However, if $\widetilde{K}$ is an arbitrary compact set of X, it is clear that there exists n_0

such that $\tilde{K} \subset Y_{\nu_0+n}$ for all $n \geq n_0$. As $g_{n_0} \in \mathcal{H}(Y_{\nu_0+(n_0+1)})$, we have $g_{n_0} \in \mathcal{H}(\tilde{K})$, so $\sup_{\tilde{K}} |g_{n_0}| \leq \widetilde{C_{\tilde{K}}}$. Moreover, using our construction, we observe that for all $n \geq n_0$, we have:

$$\sup_{\tilde{K}} |g_n| \leq \sup_{\tilde{K}} |g_n - g_{n_0}| + \sup_{\tilde{K}} |g_{n_0}| \leq \sup_{\overline{Y_{\nu_0+n}}} |g_n - g_{n_0}| + \sup_{\tilde{K}} |g_{n_0}| \leq \varepsilon + \widetilde{C_{\tilde{K}}} = C_{\tilde{K}}.$$

Applying Montel's theorem, there exists a subsequence $\{g_{n_k}\}_k$ which converges uniformly on every compact subset of X. And by Weierstrass' theorem:

$$g := \lim_{k\to\infty} g_{n_k} \in \mathcal{H}(X).$$

Finally, note that:

$$\sup_{z\in K} |(f-g)(z)| \leq \sup_{z\in K} |(f-g_0)(z)| + \sup_{z\in K} |(g_0-g)(z)| < \varepsilon.$$

Therefore, every holomorphic function in a neighborhood of K can be uniformly approximated on K by holomorphic functions on X, as desired.

Remark. Theorem 3.4 becomes the classical Runge's theorem when:

i) $X = \hat{\mathbb{C}} \equiv \mathbb{C} \cup \{\infty\}$ is the Riemann sphere and $K \subset \mathbb{C}$,

ii) $q_\infty = \infty$ for the component W_∞ of $X \backslash K$ containing ∞.

We now turn our attention to the proof of theorem 3.4. For this purpose, we shall follow the same procedure that we have detailed in the case of the Riemann sphere. We will only focus on the details of the challenges posed by the new construction.

The two new problems that arise when we try to generalize our result are:

1. the existence of Green's functions in Riemann surfaces,
2. estimates for the solution to the $\bar{\partial}$-equation in Riemann surfaces.

Obtaining a subharmonic weight function from the Green's functions of the holes can be achieved in a simple way.

1. Existence of Green's functions in Riemann surfaces. There is not always a Green's function for a Riemann surface. From the viewpoint of potential theory a Riemann surface can be classified as:

(1) hyperbolic, if it has a non-constant bounded subharmonic function,

(2) elliptic, if it is compact, or

(3) parabolic, otherwise.

We call this classification potential-theoretic because the condition of having a bounded subharmonic function is equivalent to the existence of a Green's function.

Given this, it is necessary to impose that the holes W_i are hyperbolic. This follows from the fact that holes are regular domains, see Figure 3.

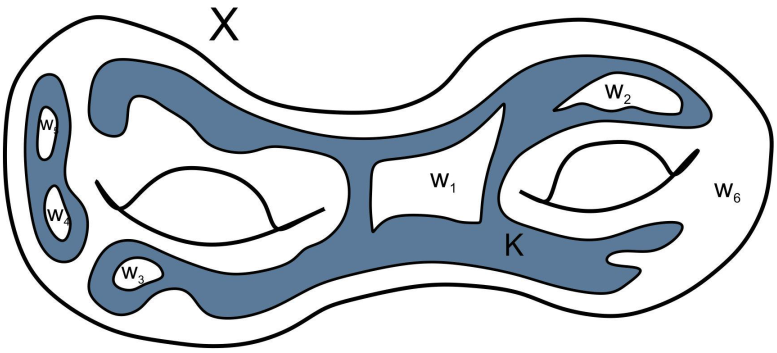

Figure 3: Example of our setting.

Proposition 3.7. [6, pp. 95] *Let $\xi \in \partial\Omega$. If the connected component of $\partial\Omega$ containing ξ consists of more that one point, then ξ is a regular point for Ω. In particular, if Ω is simply connected, then every point of $\partial\Omega$ is a regular point.*

Proposition 3.8. [14, pp. 118] *Let X be a Riemann surface and let $\Omega \subset X$ be an open subset all of whose boundary points are regular. Then the Dirichlet problem has a solution on Ω.*

We have that $\Omega = W_i$ is hyperbolic. Let us give now the definition of Green's function.

Definition 3.9. Let M be a Riemann surface. A Green's function for M is a map $g_M : M \times M \longrightarrow (-\infty, \infty]$ such that for each $x \in M$:

(a) $g_M(\cdot, x)$ is harmonic on $M \backslash \{x\}$ (superharmonic on M),

(b) if z is any local coordinate in a neighborhood U of x which $z(x) = 0$ then

$$g_M(\cdot, x) - \log \frac{1}{|z(\cdot)|}$$

is harmonic on U,

(c) if H is any other superharmonic function satisfying (a) and (b) then

$$g_M(\cdot, x) \leq H(\cdot).$$

In our case, identifying M with W_i and x with q_i,

$$G_i(z) := g_{W_i}(z, q_i) \quad z \in W_i.$$

Remark. On a Riemann surface M with boundary, a Green's function is a solution of the distributional boundary value problem

$$i\partial\bar{\partial} g_M(\cdot, x) = 2\pi\delta_x \qquad g_M(\cdot, x)\big|_{\partial M} = 0$$

as x varies over the points of the interior of M.

2. $\bar{\partial}$-estimates in Riemann surfaces. The version of Hörmander's theorem that we want to use involves properties of holomorphic line bundles over Riemann surfaces. It will be detailed further on, in the statement of theorem 3.10. For a more through approach to these we refer the reader to [4, 14].

Now we shall see that we are under the conditions of theorem 3.10. Fix a Hermitian metric

$$\omega = e^{-\psi} \frac{i}{2} dz \wedge d\bar{z}$$

for the compact Riemann surface X and note that a Riemannian metric for X is a Hermitian metric for $T_X^{1,0} \equiv (K_X)^*$, where $T_X^{1,0}$ and K_X are the holomorphic tangent and canonical bundles of X, respectively. As every Riemann surface admits a line bundle that has a metric of strictly positive curvature $c(\cdot)$, there is a holomorphic line bundle $L \longrightarrow X$ and a smooth Hermitian metric $e^{-\varphi}$ for L such that $i\partial\bar{\partial}\varphi$ is a strictly positive $(1,1)$-form. This means that

$$ic(\varphi) = i\partial\bar{\partial}(\varphi) = l\omega,$$

with l a strictly positive function.

We note that if F and F' are line bundles over X, we can form a new line bundle $F \otimes F'$ by taking tensor products on the fibers. Moreover if ϕ is a metric on F and ϕ' is a metric on F' then $\phi + \phi'$ is a metric on $F \otimes F' \equiv F + F'$.

Thus, the holomorphic line bundle $L \longrightarrow X$ with hermitian metric $e^{-\varphi}$ is strictly positive. Now we modify this line bundle $L \longrightarrow X$ to achieve our goal. For this purpose we introduce a new free-parameter $k >> 0$, which we will later establish, and we consider $L^{\otimes k} = kL$ the product of L with itself k times. As the metric on L is represented by a smooth function φ, then the metric on $L^{\otimes k}$ is given by $k\varphi$.

Now we consider the Picard group $\mathrm{Pic}(X)$ of holomorphic line bundles on a complex manifold X. We have $L^{\otimes k} \longrightarrow X$ a holomorphic line bundle with Hermitian metric $e^{-k\varphi}$, and it is clear that $i\partial\bar{\partial}(k\varphi)$ is a strictly positive $(1,1)$-form. More precisely

$$ic(k\varphi) = i\partial\bar{\partial}(k\varphi) = k\, i\partial\bar{\partial}(\varphi) = k\, l\omega,$$

with l a strictly positive function. However, we find a technical difficulty: Hörmander's estimate for the $\bar{\partial}$-equation deals with $(1,1)$-forms rather than $(0,1)$-forms. We can always twist the line bundle $L^{\otimes k}$ with the canonical bundle to shift from $(0,1)$-forms to $(1,1)$-forms. The bundle $L^{\otimes k}$ can be expressed as $L^{\otimes k} = K_X + F_k$ where K_X is the canonical line bundle and

$$F_k = L^{\otimes k} - K_X = L^{\otimes k} + (K_X)^* = L^{\otimes k} + T_X^{1,0}.$$

As we have $L^{\otimes k}$ with the metric $k\varphi$ and $T_X^{1,0}$ with the metric inherited from the Hermitian metric on X, then the metric on F_k is

$$k\varphi + \psi.$$

Thus, we obtain

$$\begin{aligned} \text{section on } L^{\otimes k} \quad &\longleftrightarrow \quad (1,0) \text{ section on } F_k \\ (0,1)\text{-form on } L^{\otimes k} \quad &\longleftrightarrow \quad (1,1)\text{-form valued on } F_k. \end{aligned}$$

Since X is a compact Riemann surface, taking $k >> 0$ big enough we see that the metric of F_k is strictly positive. This means:

$$ic(k\varphi + \psi) = i\partial\bar{\partial}(k\varphi + \psi) = \tilde{g}\omega$$

with $\tilde{g}$ a strictly positive function. The key is that φ is strictly positive and X is a compact Riemann surface.

We have finished the first part of the proof. In the next one, we will focus on the open (non-compact) Riemann surface $X \backslash \{q_1, \dots, q_n\}$, where all the results of the first part also apply.

The following step is to modify the metric $k\varphi$ of $L^{\otimes k}$ so that the problem can be solved. In order to do so, we will use the following fact: if $k\varphi$ is a metric on $L^{\otimes k}$, then any other metric on $L^{\otimes k}$ can be written as $k\varphi + \Upsilon$ where Υ is a function. In our case, $\Upsilon \equiv M\phi$ where $M >> 0$ is a free-parameter and ϕ is the subharmonic function in $X \backslash \{q_1, \dots, q_n\}$ given by the Green's function in the holes. We have

$$ic(M\phi) = i\partial\bar{\partial}(M\phi) = M\, i\partial\bar{\partial}(\phi) = Ml'\omega$$

with l' a non-negative function.

Thus, the metric $k\varphi + M\phi$ of $L^{\otimes k}$ is strictly positive in $X \backslash \{q_1, \dots, q_n\}$. Then

$$ic(k\varphi + M\phi) = i\partial\bar{\partial}(k\varphi + M\phi) = k\, i\partial\bar{\partial}(\varphi) + M\, i\partial\bar{\partial}(\phi) = (kl + Ml')\omega$$

with l a strictly positive and l' a non-negative function.

Therefore, $k\varphi + M\phi + \psi$ is also a strictly positive metric on F_k. This means that

$$ic(k\varphi + M\phi + \psi) = i\partial\bar{\partial}(k\varphi + M\phi + \psi) = g\omega$$

with g a strictly positive function.

We use a more general version of Hörmander's theorem for complete Kähler manifolds –and in particular for Stein manifolds– which we can find in [4]. Here, we use that a connected Riemann surface is a Stein manifold if and only if it is open (not-compact). As X is a compact Riemann surface, then $X \backslash \{q_1, \dots, q_n\}$ is an open Riemann surface and therefore $X \backslash \{q_1, \dots, q_n\}$is a Stein manifold.

Theorem 3.10. [4, pp. 38] *Let F be a holomorphic line bundle endowed with a metric Φ over a Riemann surface M which has some complete Kähler metric. Assume the metric Φ on F has (strictly) positive curvature and that $ic(\Phi) = i\partial\bar{\partial}(\Phi) = g\omega$, with g a strictly positive function and ω is a Kähler metric on M.*

Let α be a $\bar{\partial}$-closed $(1,1)$-form with values on F. Then there is a $(1,0)$-form u with values on F such that:

$$\bar{\partial}u = \alpha \qquad \textit{and} \qquad ||u||^2 \leq \frac{1}{g}\,||\alpha||^2,$$

provided the right hand side is finite.

We must stress that we do not need to assume that the Kähler metric appearing in the final estimate is complete, only that the manifold has some complete metric.

Note that if $\alpha = s\xi \otimes d\bar{z}$, then we have

$$||u||^2 = \int_M |u|^2 e^{-\Phi}\omega \leq \frac{1}{g}\int_M |s|^2 e^{-\Phi}\frac{i}{2}\, dz \wedge d\bar{z} = \frac{1}{g}\,||\alpha||^2.$$

Remark. Set

$$M \equiv X \backslash \{q_1, \dots, q_n\},$$
$$F \equiv F_k = L^{\otimes k} + T_X^{1,0}, \qquad L \equiv L^{\otimes k},$$
$$\Phi \equiv k\varphi + M\phi + \psi, \qquad \alpha \equiv (f\bar{\partial}\chi)\xi \otimes d\bar{z}, \qquad s \equiv f\bar{\partial}\chi.$$

Then there is the correspondence

$(1,1)$-form with values on $F \equiv L + T_M^{1,0}$ $\longleftrightarrow$ $(0,1)$-form with values on L

$(1,0)$-form with values on $F \equiv L + T_M^{1,0}$ $\longleftrightarrow$ $(0,0)$-form (function) with values on L.

In particular, we have the following estimate:

$$\int_{X\setminus\{q_1,\dots,q_n\}} |u|^2 e^{-(k\varphi+M\phi)}\omega \leq \frac{1}{g}\int_{X\setminus\{q_1,\dots,q_n\}} |f\bar{\partial}\chi|^2 e^{-(k\varphi+M\phi)}\frac{i}{2}\,dz\wedge d\bar{z}.$$

Since φ is a smooth function in X, which is compact, φ is bounded above and below in X and, in particular, in $X\setminus\{q_1,\dots,q_n\}$. Thus there exist C_1, C_2 such that $C_1 \leq e^{-k\varphi} \leq C_2$, and we get a new estimate that is similar to the one we obtained for the Riemann sphere:

$$C_1\int_{X\setminus\{q_1,\dots,q_n\}} |u|^2 e^{-M\phi}\omega \leq \frac{C_2}{g}\int_{X\setminus\{q_1,\dots,q_n\}} |f\bar{\partial}\chi|^2 e^{-M\phi}\frac{i}{2}\,dz\wedge d\bar{z}.$$

On the other hand $\mathrm{Supp}(\bar{\partial}\chi) \subset \bigcup_{i=1}^n \{z\in W_i : \delta - t\delta/2 \leq G_i(z)\leq \delta\}$, and there exists $\lambda > 0$ such that

$$i\partial\bar{\partial}(M\phi) \geq M\lambda\,\omega$$

and so

$$g \geq M\lambda,$$

both on Supp $(\bar{\partial}\chi)$. Therefore:

$$\begin{aligned}
C_1\int_{X\setminus\{q_1,\dots,q_n\}} |u|^2 e^{-M\phi}\omega &\leq \frac{C_2}{g}\int_{X\setminus\{q_1,\dots,q_n\}} |f\bar{\partial}\chi|^2 e^{-M\phi}\frac{i}{2}\,dz\wedge d\bar{z}\\
&\leq \frac{C_2}{M\lambda}\int_{\mathrm{Supp}(\bar{\partial}\chi)} |f\bar{\partial}\chi|^2 e^{-M\phi}\frac{i}{2}\,dz\wedge d\bar{z}\\
&\leq \frac{C_2}{M\lambda}\int_{\cup_i\{z\in W_i:\delta-t\delta/2\leq G_i(z)\leq\delta\}} |f\bar{\partial}\chi|^2 e^{-M\phi}\frac{i}{2}\,dz\wedge d\bar{z}.
\end{aligned}$$

A final remark: this proof does not work for $\mathbb{C}^n$ and therefore it cannot be generalized for n-dimensional complex manifolds with $n > 1$. This is so because one of the main tools of our method are Green's functions, which are subharmonic but not plurisubharmonic, as we would require in the case of several variables.

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