On computing flat outputs through Goursat normal form

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Resum *(CAT)*
Aquest article estudia el càlcul de les sortides planes mitjançant la forma normal de Goursat del sistema de Pfaff associat a qualsevol sistema de control en variables d'estat. L'algorisme consta de tres passos: i) transformació del sistema de control en el seu sistema de Pfaff equivalent; ii) càlcul de la forma normal de Goursat; iii) reescriptura de les equacions en les noves variables d'estat. Aquí, una realimentació simplifica les equacions i, per tant, les sortides planes es calculen de manera senzilla. L'algorisme s'aplica a un vehicle amb rodes extensibles. Gràcies a la propietat de planitud diferencial, s'obtenen les trajectòries entre dos punts donats.

Abstract *(ENG)*
This paper is devoted to computation of flat outputs by means of the Goursat normal form of the Pfaffian system associated to any control system in state space form. The algorithm consists of three steps: i) transformation of the system into its Pfaffian equivalent; ii) computation of the Goursat normal form; iii) rewriting of the state space equations in the new variables. Here, a feedback law simplifies the equations and, therefore, the flat outputs can be easily computed. The algorithm is applied to a car with expanding wheels. Point to point trajectories are obtained thanks to the property of differential flatness.

1. Introduction

Since 1980, the problem of feedback linearization for nonlinear control systems has been considered by different authors in several frameworks. Different routes to linearization have been traced; namely, linearization by static feedback; linearization by prolongations; linearization by dynamic feedback, and finally flatness. The main mathematical tool to study these problems is differential geometry. Notions such as Lie brackets and involutive fields or distributions, which can be found in basic books of nonlinear control theory [9, 14], have been followed by differential forms and Pfaffian systems [3, 15].

Differential flatness was introduced in the 90's by Michel Fliess and coworkers [5]. A differentially flat nonlinear system can perform any point-to-point desired trajectory. Other systems do not hold this property. Differentially flat systems are dinamically feedback equivalents to linear systems based on chains of integrators. Initial and final conditions are transferred, through diffeomorphism, to the equivalent linear system where the required inputs are designed. Inputs of the nonlinear system are obtained by application of the diffeomorphism and the feedback law.

Unfortunately, necessary and sufficient conditions to check flatness for a general nonlinear system do not exist. Since mid nineties, extensive work has been done in this direction, but only some particular cases have been solved [6, 11, 12].

Control systems are usually presented in state space form. In this paper, we convert state space form control systems into their equivalent Pfaffian systems [3, 15]. A Pfaffian system consists in a set of independent one forms. These one forms are written in the Goursat normal form which, when transformed again in state space equations, become very simple equations by addition of a feedback law and, hence, allow to find the flat outputs in an easily manner.

This paper is organized as follows: Section 2 contains a summary on how to compute Goursat normal forms for a set of independent one forms, as well as a brief introduction to nonlinear control systems. In Section 3 the relationship between control systems in state space form and its equivalent Pfaffian system is explained. The main contribution of this paper is the link between the Goursat normal form and the computation of the flat outputs. The inclusion of a feedback law plays a crucial role in this sense. Details on how to compute the flat outputs once the Goursat normal form is achieved are also explained. Section 4 is devoted to illustrate the whole process through an example, which corresponds to a system with expanding wheels [1]. Simulations are given in Section 5, where an additional control law is applied to overcome errors in the initial conditions. The paper ends with the conclusions. A reduced version of this paper has been accepted for publication at European Control Conference 2014 [7].

2. Background

2.1 Normal form for differential one forms

This section provides a very brief summary on how to compute normal forms for differential one forms. A detailed approach can be found, for example, in [3, 15]. In the sequel, all the vector fields an differential forms are supposed to be $\mathcal{C}^\infty$.

Definition 2.1. A system of the form

$$\alpha_1 = \alpha_2 = \cdots = \alpha_s = 0,$$

where the α_i are independent 1-forms on an n-dimensional manifold, is called a *Pfaffian system*.

Definition 2.2. A *smooth codistribution* smoothly associates a subspace of the cotangent space at each point $p \in M$.

Definition 2.3. The sequence of decreasing codistributions

$$I^{(k)} \subset I^{(k-1)} \subset \cdots \subset I^{(1)} \subset I^{(0)}$$

is called the *derived flag* of $I^{(0)}$, where

$$I^{(k+1)} = \{\lambda \in I^{(k)} : d\lambda \equiv 0 \mod I^{(k)}\}.$$

Definition 2.4. Let $\alpha \in \Omega^1(M)$. The integer r defined by $(d\alpha)^r \wedge \alpha \neq 0$ and $(d\alpha)^{r+1} \wedge \alpha = 0$ is called *rank* of α.

We are interested in transforming the generators of Pfaffian systems into a normal form by means of a coordinate transformation. Let us study first Pfaffian systems of codimension 1, or systems consisting of a single equation $\alpha = 0$. The following theorem allows us, under a rank condition, to write α in a normal form.

Theorem 2.5 (Pfaff Theorem). *Let $\alpha \in \Omega^1(M)$ have constant rank r in a neighborhood of p. Then, there exists a coordinate chart (U, z) such that, in these coordinates,*

$$\alpha = dz_1 + z_2 dz_3 + \cdots + z_{2r} dz_{2r+1}.$$

The proof is constructive and is based on finding functions $f_1, \ldots, f_{r+1}$ and $g_1, \ldots, g_r$ ($2r+1 < n$, where $\dim M = n$) such that

$$(d\alpha)^r \wedge \alpha \wedge df_1 = 0,$$
$$(d\alpha)^{r-1} \wedge \alpha \wedge df_1 \wedge df_2 = 0,$$

up to f_r,

$$d\alpha \wedge \alpha \wedge df_1 \wedge df_2 \wedge \cdots \wedge df_r = 0,$$
$$\alpha \wedge df_1 \wedge df_2 \wedge \cdots \wedge df_r \neq 0,$$

so that,

$$\alpha = df_{r+1} + g_1 df_1 + \cdots + g_r df_r.$$

A new set of variables, diffeomorphic to the state space variables, is defined as follows:

$$z_1 = f_{r+1}, \quad z_{2i} = g_i, \quad z_{2i+1} = f_i,$$

with $1 \leq i \leq r$.

For Pfaffian systems of codimension two, a particular case is given by Pfaffian system with four variables. The algorithm to transform the one forms into a canonical form is obtained in Engel's theorem:

Theorem 2.6 (Engel's Theorem). *Let I be a dimension two codistribution, spanned by $I = \langle \alpha_1, \alpha_2 \rangle$ of four variables. Setting $I^{(0)} = I$, if the derived flag satisfies*

$$\dim I^{(1)} = 1,$$
$$\dim I^{(2)} = 0,$$

then there exist coordinates z_1, z_2, z_3, z_4 such that

$$I = \{dz_4 - z_3 dz_1, dz_3 - z_2 dz_1\}.$$

The proof is also constructive and uses the previous theorem.

Engel's theorem can be generalized to a system with n configuration variables and $n-2$ constraints. The following theorem states the conditions required in order to convert a Pfaffian system into its Goursat normal form.

Theorem 2.7 (**Goursat Normal Form**). *Let I be a Pfaffian system spanned by s 1-forms, $I = \{\alpha_1, \dots, \alpha_s\}$, on a space of dimension $n = s + 2$, such that*

$$d\alpha_s \not\equiv 0 \quad \mod I.$$

Assume also that there exists an exact form π, with $\pi \neq 0 \mod I$, satisfying the Goursat congruences

$$d\alpha_i \equiv -\alpha_{i+1} \wedge \pi \quad \mod \alpha_1, \dots, \alpha_i, \quad 1 \leq i \leq s - 1.$$

Then there exists a coordinate system $z_1, z_2, \dots, z_n$ in which the Pfaffian system is in Goursat normal form,

$$I = \{dz_3 - z_2 dz_1, dz_4 - z_3 dz_1, \dots, dz_n - z_{n-1} dz_1\}.$$

Finally, in order to study Pfaffian systems of codimension greater than two, we will use the extended Goursat normal form. That is, a Pfaffian system of codimension $m+1$ and generated by n constraints of the form

$$I = \{dz_i^j - z_{i-1}^j dz_0 : i = 1, \dots, s_j; j = 1, \dots, m\}.$$

Conditions to convert a Pfaffian system into the extended Goursat normal form are given in the following theorem:

Theorem 2.8 (**Extended Goursat Normal Form**). *Let I be a Pfaffian system of codimension $m+1$ in $\mathbb{R}^{n+m+1}$. The system can be put into the extended Goursat normal form if, and only if, there exists a set of generators $\{\alpha_i^j : i = 1, \dots, s_j; j = 1, \dots, m\}$ for I and an exact one-form π such that, for all j,*

$$
\begin{aligned}
d\alpha_i^j &\equiv -\alpha_{i+1}^j \wedge \pi \quad && \mod I^{(s_j - i)}, \quad i = 1, \dots, s_j - 1, \\
d\alpha_i^j &\not\equiv 0 \quad && \mod I.
\end{aligned}
$$

All the proofs of these theorems are constructive and are outlined in [3, 15].

2.2 Feedback linearization of control systems

Definition 2.9. A nonlinear control system

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i \quad x \in \mathbb{R}^n \tag{1}$$

is said to be *static feedback linearizable* if it is possible to find a feedback

$$u = \alpha(z) + \beta(z) v, \quad u \in \mathbb{R}^m, \quad v \in \mathbb{R}^m, \quad z \in \mathbb{R}^n,$$

and a local diffeomorphism

$$z = \phi(x)$$

such that the original system is transformed into a linear controllable system

$$\dot{z} = Az + Bv,$$

where A and B are matrices of appropriate size.

Necessary and sufficient conditions to check static feedback linearization were given in [8, 10]. A generalization of the static feedback linearization is a dynamic feedback transformation [4].

Definition 2.10. A nonlinear system

$$\dot{x} = f(x, u), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \tag{2}$$

is said to be *dynamic feedback linearizable* if there exists:

1. A regular dynamic compensator

$$\begin{cases} \dot{z} = a(x, z, v) \\ u = b(x, z, v) \end{cases} \tag{3}$$

   with $z \in \mathbb{R}^q$ and $v \in \mathbb{R}^m$. The regularity assumption implies the invertibility of (3) with input v and output u.

2. A local diffeomorphism

$$\psi = \Psi(x, z) \tag{4}$$

   with $\psi \in \mathbb{R}^{n+q}$, such that the original system (2) with the dynamic compensator (3), after applying (4), becomes a constant linear controllable system:

$$\dot{\psi} = A\psi + BV.$$

A system is dynamic feedback linearizable if, and only if, it is differentially flat. Differential flatness was introduced by M. Fliess and coworkers in [5].

Definition 2.11. Let (1) be a nonlinear system with m inputs. Roughly speaking, this system is differentially flat if there exist m functions $(y_1, \ldots, y_m)$, equal in number to the number of inputs, such that:

1. Each variable y_i is a function of the states, the inputs, and a finite number of the inputs derivatives.

2. The states and the inputs can be expressed as functions of the variables $(y_1, \ldots, y_m)$ and their derivatives up to a certain order.

The variables $(y_1, \ldots, y_m)$ are called *flat outputs*.

The relationship between Goursat normal form of Pfaffian systems and nonholonomic [13] control systems in state space form is as follows. Given a two input driftless system

$$\dot{x} = g_1 u_1 + g_2 u_2, \quad x \in \mathbb{R}^n,$$

in state space form, its equivalent Pfaffian system can be obtained by finding $n - 2$ one forms α_i, such that $\alpha_i \lrcorner g_1 = 0$ and $\alpha_i \lrcorner g_2 = 0$ for all $i = 1, \ldots, n - 2$.

By applying one of the above theorems, the Goursat normal form can be found. As explained above, this includes the definition of a new set of state variables $z_1, \ldots, z_n$. The dynamics associated to the system in these new variables is got by differentiation of each of these variables, which leads to

$$\dot{z} = g_1(z)u_1 + g_2(z)u_2.$$

Finally, the above system can be expressed as

$$
\begin{aligned}
\dot{z}_1 &= \overline{u}_1, \\
\dot{z}_2 &= \overline{u}_2, \\
\dot{z}_3 &= z_2\overline{u}_1, \\
&\vdots \\
\dot{z}_n &= z_{n-1}\overline{u}_1,
\end{aligned}
\tag{5}
$$

by application of a feedback law. A similar algorithm can be applied to any nonholonomic system. The structure of system (5) is very convenient in order to find the flat outputs.

3. Algorithm to find flat outputs

Consider the system given by

$$
\dot{x} = \sum_{i=1}^{\overline{m}} g_i u_i, \quad x \in \mathbb{R}^{\overline{n}},
$$

where $\overline{m}$ is the number of controls and $\overline{n}$ the dimension of state space.

First of all, an equivalent formulation of the system in differential forms will be given. In order to achieve this goal, $\overline{n} - \overline{m}$ differential forms that annihilate the control vector fields must be found. Then, the Pfaffian system consists in $\overline{n} - \overline{m}$ equations:

$$
\omega_1 = \omega_2 = \ldots = \omega_{\overline{n}-\overline{m}} = 0,
$$

where $\omega_i \in \langle g_1, \ldots, g_{\overline{m}} \rangle^{\perp}$, $i = 1, \ldots, \overline{n} - \overline{m}$. Given a Pfaffian system in $\mathbb{R}^{n+m+1}$, where $\overline{n} = n+m+1$ and $\overline{m} = m+1$ is the transforming system codimension, these forms are expressed in their extended Goursat canonical form

$$
I = \{\omega_i^j = dz_i^j - z_{i+1}^j dz_0 : i = 1, \ldots, s_j, \ j = 1, \ldots, m\},
$$

where s_j satisfies that $\overline{n} = m + 1 + \sum_{j=1}^m s_j$.

At this point, the goal is to rewrite the system using vector fields. In order to do this, we must find $m + 1$ vector fields that vanish on the ideal of forms, i.e., vector fields expressed in a generic form for $k = 0, \ldots, m$ as

$$
\overline{g}_k = \left(a_0, a_1^1, \ldots, a_{s_1}^1, a_{s_1+1}^1, \ldots, a_1^m, \ldots, a_{s_m}^m, a_{s_m+1}^m\right)
$$

meeting the following conditions:

$$
\overline{g}_k \lrcorner
\left\{
\begin{array}{c}
dz_1^j - z_2^j \, dz_0 \\
dz_2^j - z_3^j \, dz_0 \\
\vdots \\
dz_{s_j}^j - z_{s_j+1}^j \, dz_0
\end{array}
\right\} = 0, \quad j = 1, \ldots, m.
$$

A possible solution is $\overline{g}_0$ such that:

$$
\begin{aligned}
a_0 &= 1 \\
a_1^j &= z_2^j \\
&\vdots \\
a_{s_j}^j &= z_{s_j+1}^j \\
a_{s_j+1}^j &= 0
\end{aligned}
$$

and

$$\overline{g}_j = \frac{\partial}{\partial z^j_{s_j+1}}, \quad j = 1, \dots, m$$

so that, in the new variables, the system reads:

$$
\begin{cases}
\dot{z}_0 &=& u_0 \\
\dot{z}^1_1 &=& z^1_2 u_0 \\
&\vdots& \\
\dot{z}^1_{s_1} &=& z^1_{s_1+1} u_0 \\
\dot{z}^1_{s_1+1} &=& u_1 \\
\dot{z}^2_1 &=& z^2_2 u_0 \\
&\vdots& \\
\dot{z}^2_{s_2} &=& z^2_{s_2+1} u_0 \\
\dot{z}^2_{s_2+1} &=& u_2 \\
&\vdots& \\
\dot{z}^m_1 &=& z^m_2 u_0 \\
&\vdots& \\
\dot{z}^m_{s_m} &=& z^m_{s_m+1} u_0 \\
\dot{z}^m_{s_m+1} &=& u_m.
\end{cases}
\tag{6}
$$

Remark. The system obtained by application of the above algorithm and the system obtained by differentiation of the system variables $\left\{ z^j_i, \ i = 1, \dots, s_j, \ j = 1, \dots, m \right\}$ can be different. To get the same system a feedback law must be included.

From equations (6), it is straightforward to obtain the flat outputs. Consider $y_0 = z_0$ and $y_1 = z^1_1$ as the first flat outputs. Then, $z^1_2, \dots, z^1_{s_1+1}$, can be expressed in terms of y_0, y_1 and its derivatives, dividing both sides by u_0. The same happens for the remaining equation blocks. Therefore, the flat outputs are

$$
\begin{aligned}
y_0 &=& z_0, \\
y_1 &=& z^1_1, \\
&\vdots& \\
y_m &=& z^m_1,
\end{aligned}
$$

and the remaining variables are expressed as:

$$
\begin{cases}
z^j_2 = \dot{z}^j_1 / u_0 = \dot{y}_j / \dot{y}_0, \\
z^j_3 = \dot{z}^j_2 / u_0 = z^j_3 \left(\dot{y}_0, \ddot{y}_0, \dot{y}_j, \ddot{y}_j \right) \\
\vdots \\
z^j_{s_j+1} = \dot{z}^j_{s_j} / u_0 = z^j_{s_j+1} \left(\dot{y}_0, \dots, y_0^{(s_j)}, \dot{y}_j, \dots, y_j^{(s_j)} \right).
\end{cases}
$$

Considering $s_0 = \max\{s_1, \dots, s_m\}$, we need $s_0 + \overline{n}$ variables to describe $\overline{n}$ variables. So that, the system has to be prolonged as follows:

$$
\begin{aligned}
z^0_1 &=& u_0, \\
&\vdots& \\
z^0_{s_0} &=& u_0^{(s_0-1)}, \\
v &=& \dot{z}^0_{s_0}.
\end{aligned}
$$

So far we have two well defined diffeomorphisms: one between the original state variables $(x_1, \ldots, x_{\overline{n}})$ and the new state variables $(z_1, \ldots, z_{\overline{n}})$ and the second one between $(z_1, \ldots, z_{\overline{n}})$ and the flat outputs and their derivatives. So that, given a set of initial and final conditions for the original system, these conditions are mapped into system (6) through the diffeomorphism. These conditions, plus additional conditions for the extended variables, are transferred to initial and final conditions for the flat outputs by using the second diffeomorphism.

Given $2(s_j + 1)$, $j = 0, \ldots, m$, initial and final conditions for each flat output and its derivatives, there exists a unique $2s_j + 1$ degree polynomial that meets these conditions. Once the polynomial has been defined, the controls $u_j(t)$, $j = 0, \ldots, m$, must be found from these equations:

$$w_0 = y_0^{(s_0+1)} = v,$$
$$w_j = y_j^{(s_j+1)} = \frac{\mathrm{d}^{s_j+1}}{\mathrm{d}t^{s_j+1}} z_1^j, \quad j = 1, \ldots, m.$$

Control laws for the original system are found by mapping back the control laws trough the feedback transformation.

4. Example

Consider the system corresponding to a vehicle with equal and expanding back wheels and equal front wheels with a fixed radius l, that was studied in [1, 2]. The vehicle dynamics is described by

$$\begin{pmatrix} \dot{r} \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -(\tan\alpha)/l \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 1 \\ r/l \end{pmatrix} u_2 = f_1 u_1 + f_2 u_2, \tag{7}$$

where θ_1 and θ_2 are, respectively, the variables defining the angular position of the front and rear wheels, α is a constant corresponding to the angle between the horizontal and the line obtained joining the wheel centers, and r is the radius of the back wheels that varies with time. A diagram of the system is plotted in Fig. (1).
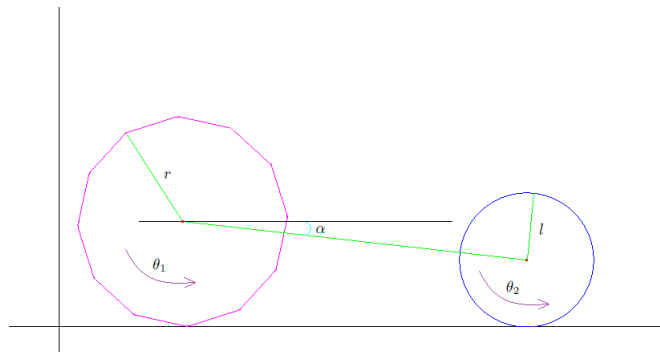

Figure 1: System diagram.

The codistribution defined as

$$I^{(0)} = \Delta^\perp = \{\omega \in \Lambda^1 \mid f_i \lrcorner \omega = 0, \ \forall f_i \in \triangle\}$$

has to be found. A possible solution is

$$\omega = \tan \alpha \, dr - r \, d\theta_1 + l \, d\theta_2.$$

The goal is to put ω into the Goursat normal form. This one form fulfills $(d\omega) \wedge \omega \neq 0$ and $(d\omega)^2 \wedge \omega = 0$, so $\text{rang}(\omega) = 1$ and Pfaff theorem can be applied. First of all, a function f_1 such that

$$d\omega \wedge \omega \wedge df_1 = 0$$

has to be found. Actually this is a degree four form in a three dimensional space. Hence, it vanishes everywhere and any f_1 function works out. For simplicity, we choose

$$f_1(r, \theta_1, \theta_2) = r.$$

A second function f_2 has to satisfy

$$\omega \wedge df_1 \wedge df_2 = 0,$$
$$df_1 \wedge df_2 \neq 0.$$

Note that this is a degree three form in a three dimensional space. A possible function could be

$$f_2(r, \theta_1, \theta_2) = \theta_2 l - \theta_1 r,$$

so that,

$$\omega = df_2 + g_1 \, df_1 = dz_3 - z_2 \, dz_1.$$

The new variables expressed in terms of the original ones are

$$z_1 = r,$$
$$z_2 = -\theta_1 - \tan \alpha,$$
$$z_3 = \theta_2 l - \theta_1 r.$$

And the system expressed in the new variables is

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ z_2 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} u_2. \tag{8}$$

Note that system (8) is not in the Goursat canonical form. As remarked before, a feedback law must be applied in order to get a system like in equation (5). In this case, this feedback is

$$\overline{u}_1 := u_1,$$
$$\overline{u}_2 := -u_2.$$

Hence, system (8) becomes

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ z_2 \end{pmatrix} \overline{u}_1 + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \overline{u}_2.$$

The flat outputs are easy to obtain from this canonical form,

$$y_1 = z_3,$$
$$y_2 = z_1,$$

so that

$$\dot{y}_1 = \dot{z}_3 = z_2 \bar{u}_1,$$
$$\dot{y}_2 = \dot{z}_1 = \bar{u}_1.$$

From here we extract

$$z_2 = \frac{\dot{y}_1}{\dot{y}_2}.$$

The variables $z = (z_1, z_2, z_3)$ are expressed in terms of $y = (y_1, \dot{y}_1, y_2, \dot{y}_2)$.

In order to define a diffeomorphism between $z = (z_1, z_2, z_3)$ and $y = (y_1, \dot{y}_1, y_2, \dot{y}_2)$, the system has to be prolonged as follows

$$z_4 = \bar{u}_1,$$

and two new controls,

$$v_1 = \dot{\bar{u}}_1, \quad v_2 = \bar{u}_2,$$

are defined. Therefore, the system becomes

$$
\begin{cases}
\dot{z}_1 &= z_4 \\
\dot{z}_2 &= v_2 \\
\dot{z}_3 &= z_2 z_4 \\
\dot{z}_4 &= v_1.
\end{cases}
$$

The diffeomorphism linking the two sets of variables ($z = (z_1, z_2, z_3, z_4)$ and $y = (y_1, \dot{y}_1, y_2, \dot{y}_2)$) is

$$y_1 = z_3,$$
$$y_2 = z_1,$$
$$\dot{y}_1 = z_2 z_4,$$
$$\dot{y}_2 = z_4.$$

In the flat variables, the system reduces to a pair of second-order integrators

$$
\begin{cases}
\ddot{y}_2 = w_1 \\
\ddot{y}_1 = w_2.
\end{cases}
$$

The feedback law relating the control laws is:

$$
\begin{cases}
v_1 = w_1 \\
v_2 = (w_2 - z_2 w_1)/z_4.
\end{cases}
$$

Each flat output has to pass through four conditions (two initial conditions and two final conditions), so there exist two unique third degree polynomials such that

$$P_3(t) = y_1(t),$$
$$Q_3(t) = y_2(t).$$

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5. Simulations

A set of initial and final conditions have been chosen as follows

$$x(0) = (r(0), \theta_1(0), \theta_2(0)) = (1, \pi, \pi/2),$$
$$x(1) = (r(1), \theta_1(1), \theta_2(1)) = (2, 0, 0),$$
$$z(0) = (z_1(0), z_2(0), z_3(0)) = (1, -\pi - 1, \pi/2),$$
$$z(1) = (z_1(1), z_2(1), z_3(1)) = (2, -1, 0).$$

Adding $z_4(0) = 1$ and $z_4(1) = 3$, through the diffeomorphism, we obtain the following conditions for the flat outputs

$$y(0) = (y_1(0), \dot{y}_1(0), y_2(0), \dot{y}_2(0)) = (-\pi/2, -\pi - 1, 1, 1),$$
$$y(1) = (y_1(1), \dot{y}_1(1), y_2(1), \dot{y}_2(1)) = (0, -3, 2, 3).$$

The polynomials meeting these conditions are

$$P_3(t) = (-2\pi - 4)t^3 + (5 + 7\pi/2)t^2 + (-\pi - 1)t - \pi/2,$$
$$Q_3(t) = 2t^3 - 2t^2 + t + 1.$$

Once the polynomials have been found, the inputs are obtained by double differentiation:

$$w_2 = \frac{d^2}{dt^2} y_1 = \frac{d^2}{dt^2} P_3(t),$$

$$w_1 = \frac{d^2}{dt^2} y_2 = \frac{d^2}{dt^2} Q_3(t).$$

By applying inverse feedback, the controls $v_1(t)$ and $v_2(t)$ are found. Since $\overline{u}_2(t) = v_2(t)$ and $\dot{\overline{u}}_1(t) = v_1(t)$, the original controls can be obtained by integration:

$$u_1(t) = 1 - 4t + 6t^2,$$
$$u_2(t) = \frac{3(-2 + 4t + 4t^2 - \pi + 6t^2\pi)}{(1 - 4t + 6t^2)^2}.$$

Replacing the controls obtained in the original system, trajectories for the system variables are found through numerical integration. These trajectories are depicted in Fig.(2):
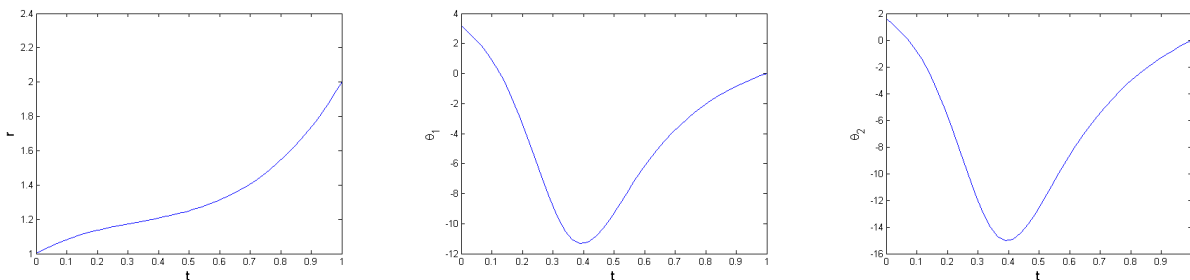

Figure 2: Behavior of system variables.

In order to circumvent errors in the initial conditions, a linear controller is added to the linear system corresponding to the flat output space. With the addition of this controller, errors in the final conditions are minimized.

The original initial and final conditions are

$$(r(0), \theta_1(0), \theta_2(0)) = (1, \pi, \pi/2),$$
$$(r(1), \theta_1(1), \theta_2(1)) = (2, 0, 0),$$

and the perturbed initial conditions are

$$(r(0), \theta_1(0), \theta_2(0)) = (1/2, 7/2, 2).$$

In the next figure, we can observe how the modified trajectories converge quickly to the desired trajectory, which is the trajectory obtained by the unperturbed initial conditions plotted in Fig. (3):
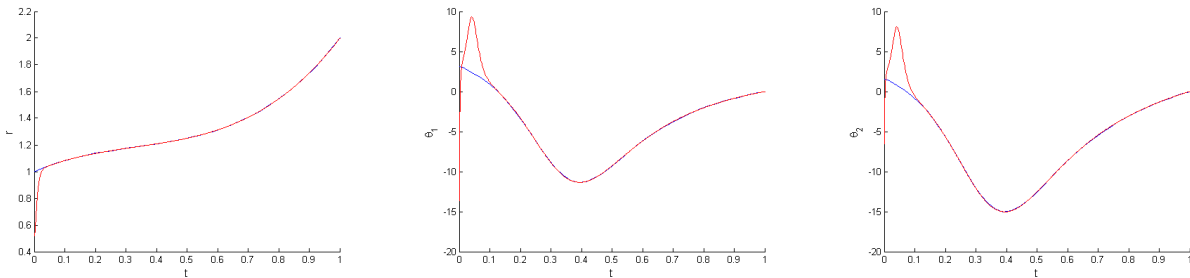

Figure 3: Trajectories of the system variables with disturbance in the initial conditions.

6. Conclusions

An algorithm to find flat outputs has been explained. This algorithm consists in finding the equivalent Pfaffian system to a control system and transforming this Pfaffian system into the Goursat normal form. This Goursat normal form, when it is written again in the state space form, is very useful in order to find the flat outputs of the system if a feedback law is included in order to simplify the equations.

As an example, point to point trajectories for a car with expanding wheels are simulated. The control laws have been obtained by transforming the system into its equivalent trivial linear system in the flat variables (chains of integrators), and designing the control laws by interpolation of the initial and final conditions. The control laws for the nonlinear system are obtained mapping back the diffeomorphisms and the feedback laws.

Future works using Goursat canonical forms include application of this algorithm to more complex control systems, as well as possible reinterpretation of existing results in the literature that have been obtained in the framework of vector fields.

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