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Revisiting the border between Newtonian mechanics and General Relativity: The periastron advance

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Summary. The problem of periastron advance, which is the basis of one of the three classical tests of relativity theory, is revised with respect to both Newtonian mechanics and General Relativity and updated in the light of recent astronomical measurements of binary pulsars. We show that in Newtonian mechanics the addition of a corrective term to Newton's law of gravitation, consistent with the principles of Newtonian mechanics, leads to the same formula of periastron advance as that used in General Relativity, which proves to be valid in all astronomical cases known, even in the cases of binary pulsars such as PSR B1913+16, PSR J1141-6545 and the so-called double pulsar PSR J0737-3039A and PSR J0737-3039B, which are considered as natural relativity laboratories. Thus, among the relativistic phenomena, the periastron advance is one that can be also understood in Newtonian terms by means of an *ad hoc* assumption.

Introduction

In the following, we review and update the problem of periastron advance in the light of recent astronomical measurements, with the aim of providing a useful academic approach in the teaching of gravity. The advance of Mercury's perihelion, which cannot be predicted in Newtonian mechanics by means of Newton's law of gravitation, is one of the three classical tests of General Relativity [1,8,13,19,21,28].

At its origin, gravitation was envisaged as an attractive force whose precise analytical formulation was subordinated to astronomical measurements which, at the time of Newton, led to the known dependence on the inverse of the distance squared. Newton himself was aware of the fact that formulations other than this one would imply a perihelion shift. The lack of evidence for that shift at that time was thus taken as a proof of validation of the aforementioned formulation [14–16].

Since the mid-19th century, as more accurate astronomical measurements became available and the advance of Mercury's perihelion was detected, several *ad hoc* proposals were made in an attempt to account for the anomalous perihelion shift of Mercury's orbit. Two alternative approaches were proposed: (1) modifying Newton's law of gravitation and (2) explaining the phenomenon as a perturbation whose ingenious origin could be, among others, the existence of a new planet, Vulcan, near the Sun; a hypothetical satellite of Mercury, solar oblateness, a ring of planets between the Sun and Mercury, or a particular

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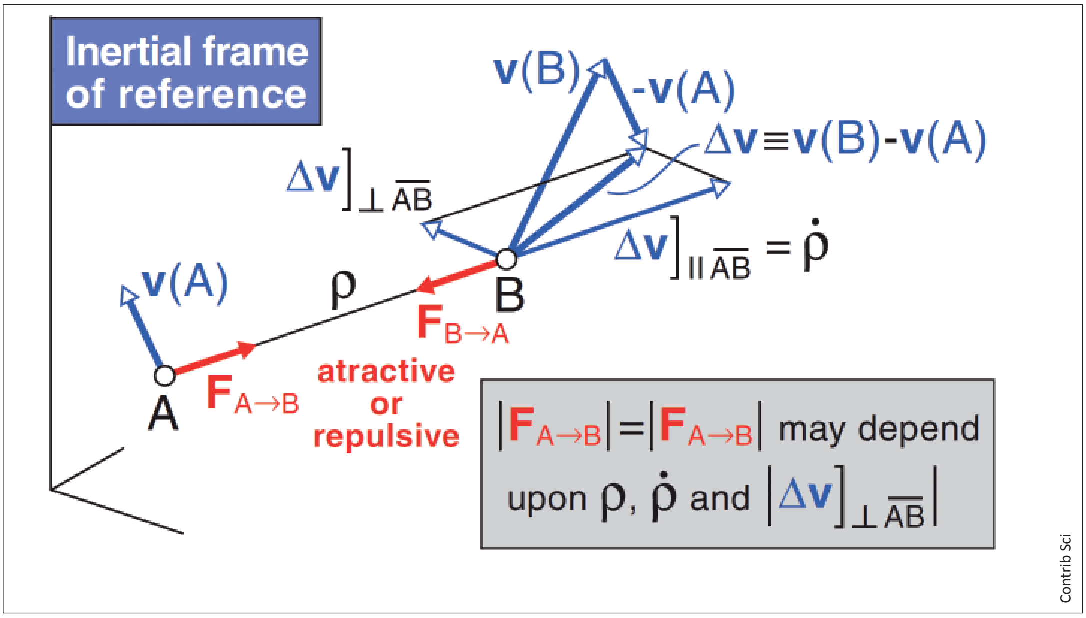

Fig. 1. Interaction forces in Newtonian mechanics.

distribution of the matter responsible for the zodiacal light [3,29]. This second alternative proved to be unsuccessful because of its incompatibility with other astronomical measurements. At the beginning of the 20th century, General Relativity accounted for the anomalous advance of Mercury's perihelion in a natural way, without any *ad hoc* assumption and without disturbing the agreement with other planetary observations.

However, in the mid-20th century, the Brans-Dicke theory of gravitation [2] appeared as an alternative to Einstein's more popular theory of General Relativity. In the Brans-Dicke theory, the reciprocal of the gravitational constant G is itself a scalar field generated by matter, which has the physical effect of changing G. The field equations contain the dimensionless constant ω, called the Brans-Dicke coupling constant, which can be chosen to fit observations. Like General Relativity, the Brans-Dicke theory predicts Mercury's perihelion advance. However, the value of ω must be very large—at least several hundred, an artificial requirement in some views—for the Brans-Dicke theory to explain the results from observations such as Mercury's perihelion advance and the radio wave deflection by the Sun. Eventually, the Brans-Dicke theory of gravitation lost relevance.

The approximations made in the context of General Relativity when calculating the periastron advance lead to a formula that can also be obtained in Newtonian mechanics, as will be shown, if a simple corrective term—consistent with the principles of Newtonian mechanics—are added to Newton's law of gravitation. The approximations leading to this formula are acceptable not only in the case of Mercury and other planets of the Solar System, but also, as will be seen, in the case of all pulsars with a measured periastron advance. In particular, PSR B1913+16, PSR J1141-6545 and the so-called double pulsar PSR J0737-3039A and PSR J0737-3039B are considered as natural relativity laboratories.

Interaction forces in Newtonian mechanics

In Newtonian mechanics, forces between two particles, A and B, are attractions or repulsions of equal modulus and thus are parallel to $\overline{AB}$. Their dependence upon position and velocity in inertial frames is restricted by space homogeneity and isotropy, by the uniformity of time, and by Galileo's principle of relativity. Accordingly, forces can only be a function of the distance ρ between the two particles, Its time derivative $\dot{\rho}$ and the modulus of the component orthogonal to $\overline{AB}$ of the difference between their velocities relatives to any inertial frame of reference (Fig. 1). Actually this last dependence is not found in the usual forces formulated in Newtonian mechanics, which are a function of just ρ and $\dot{\rho}$. However, as will be demonstrated, its introduction in the formulation of gravitation—in Newtonian mechanics the only true force between distant particles—allows a formulation that deals with the periastron advance.

Relativity corrections to Newton's law of gravitation

First-order relativity corrections are used to formulate the periastron advance per revolution. As a first step, we consider the case of a particle in a central gravitation field. The results are then extended to the two-bodies problem.

Because Einstein's field equations are prohibitively hard to solve for multi-body systems like the Solar System, an alternative approach to address the study of motion in a gravitational field was developed: Eddington, Robertson, and Shiff began to establish the "post-Newtonian" approximation of the General Relativity. Note, however, that, despite its name, "post-Newtonian" does not mean a modified Newton's law of gravitation, but rather a simplified Einstein's gravity.

The post-Newtonian formalism assumes a weak gravitational field and slow body motion—compared with the speed of light—with both conditions being fulfilled in the case of the Solar System. In this formalism, a set of parameterized correction terms are added to Newton's law to account for relativistic effects. Nordtvedt introduced up to seven parameters, which became known as the "parametrized post-Newtonian (PPN) formalism" [17,18]. In particular, the PNN formula that eventually yield the perihelion advance includes contributions from the γ (the amount of space curvature produced by one unit of mass at rest) and β (the non-linearity in the law of gravitation) PNN parameters. Taking both parameters =1 (a condition needed to be consistent with Einstein's equivalence principle), the general relativistic formula for the perihelion advance is obtained.

Particle of infinitesimal mass moving in a central gravitational field

According to General Relativity, the gravitational field in the two-body problem is described in terms of curved space-time. The field equations that describe the space-time geometry are nonlinear and the Schwarzschild metric is an exact solution to the Einstein field equations. Using the Schwarzschild coordinates, the motion of a particle of infinitesimal mass undergoing the attraction of a non-spinning free spherical mass of negligible diameter follows a path defined by the geodesics of the Schwarzschild metric [4]. In this fram, the periastron advance per revolution $\dot{\theta}$ is calculated from [5]:

$$\dot{\theta}=2\int_0^{\pi}(1-q)^{-1/2}d\omega-2\pi, \qquad (1)$$

with

$$q=\frac{2\mu(3+e\cos\omega)}{c^2a(1-e^2)}, \qquad (2)$$

where c is the vacuum velocity of light, *a* is the semimajor orbital axis, e the eccentricity, and μ = MG (G = gravitational constant, M = mass creating the field).
If q « 1, the first-order approximation of Equation (1) yields

$$\dot{\theta}=\frac{6\pi\mu}{c^2a(1-e^2)} \qquad (3)$$

This formula can be obtained from Newtonian mechanics if a corrective term consistent with Newtonian mechanics principles is added to Newton's law of gravitation.

A first-order relativity correction to Keplerian orbits can be considered as a perturbation coming from several corrective terms added to Newton's law of gravitation [6]. A set of terms describes an attractive force, while a further term describes a force—unacceptable in Newtonian mechanics—tangential to the orbit and directed towards the side of increasing radius. Their value per unit of mass is

$$F_{rad}=-\frac{\mu}{r^2}\frac{1}{c^2}\left[-\alpha\frac{2\mu}{r}+\beta 2v^2-\gamma 2\dot{r}^2-\delta\frac{\dot{r}^2}{1-\left(2\mu/rc^2\right)}\right], \qquad (4)$$

$$F_{tan}=\frac{\mu}{r^2}\frac{1}{c^2}\lambda\frac{|\dot{r}|v}{1-\left(2\mu/rc^2\right)}, \qquad (5)$$

where r is the distance to the field center, $\dot{r}$ is its time derivative, and v is the velocity. Coefficients α, β, γ, δ, and λ are dimensionless.

As $(2\mu/rc^2)$ «1, the denominator in the term of δ can be approximated as $1-(2\mu/rc^2)\approx 1$ and, consequently, Equation (4) can be written as

$$F_{rad} \approx -\frac{\mu}{r^2}\frac{1}{c^2}\left[-\alpha\frac{2\mu}{r}+\beta 2v^2-(2\gamma+\delta)\dot{r}^2\right] \quad (6)$$

Corrective terms F_{rad} and F_{tan} lead to Equation (3) of the periastron advance provided that $-\alpha+2\beta+2\lambda = 3$ [7], while parameters γ and δ can be taken arbitrarily. In Newtonian mechanics, as $\lambda=0$, condition $-\alpha+2\beta = 3$ has to be verified.

If $\alpha = -3$ and $\beta = 0$, F_{rad} reduces to

$$F_{rad} = \frac{\mu}{r^2} 6\frac{\mu}{c^2 r} \equiv \frac{\mu}{r^2} q_1 \quad (7)$$

while the reverse condition $\alpha=0$ and $\beta=3/2$ leads, with $\gamma=\delta=1$, to

$$F_{rad} = -\frac{\mu}{r^2} 3\frac{v^2-\dot{r}^2}{c^2} = -\frac{\mu}{r^2} 3\frac{v^2}{c^2} \equiv \frac{\mu}{r^2} q_2 \quad (8)$$

where v is the modulus of the velocity component orthogonal to the radius.

Any linear combination of corrective terms defined by Equations (7) and (8), with coefficients ε_1 and ε_2 verifying $\varepsilon_1 + \varepsilon_2 = 1$, defines a corrective force leading to the same periastron advance as that predicted by relativity mechanics by means of Equation (3) [22]. Certain sets of coefficients ε_1, ε_2 may be preferable if attention is paid to other phenomena.

As the corrective terms defined by Equations (7) and (8) have been obtained from perturbation theory, they must be small compared to the value μ/r^2 which they correct:

$$q_1 \equiv 6\frac{\mu}{c^2 r} << 1 \quad ; \quad q_2 \equiv 3\frac{v^2}{c^2} << 1 \quad (9)$$

Extension to the two-bodies problem

So far, a particle of infinitesimal mass moving in a central gravitation field has been considered, but an extension to the two-bodies problem can be done provided that, for two particles P_1 and P_2 with mass m_1 and m_2 respectively, the following values are used in Equations (3), (7), and (8):

$\mu = G(m_1 + m_2)$; a = semi-major axis of the relative ellipse (10)

or

$\mu = Gm_2$; $a=a_i$ (11)

where a_i is the semi-major axis of the elliptic orbit followed by P_i focusing on the system center of mass.

The case of planets of the Solar System

The maximum value of q [Equation (2)] and those of q_1 and q_2 [Equation (9)], which must be «1 in order for the approximations leading to Equation (3) in General Relativity to be acceptable, are shown in Table 1 for each planet of the Solar System.

As all q, q_1 and q_2 values are «1, the perihelion advance as calculated in General Relativity is the same as in Newtonian mechanics with the corrective term added to Newton's law of gravitation.

The case of binary pulsars

Gravitational forces much stronger than those acting in the Solar System can be found in binary systems, and hence the usual approximation made to calculate the periastron advance are questionable. Among binary systems, those with a pulsar are better known because the pulsar greatly helps in the measurement of system parameters.

The PSR B1913+16 pulsar, illustrated in Fig. 2, was the first discovered pulsar belonging to a binary system. Its discovery by Hulse and Taylor [12] in 1974 in Arecibo granted them the Nobel Prize of Physics in 1993.

With its well-known parameters [9–11,25–27], it has been considered a natural laboratory of relativistic experimentation because of the high gravitational attraction

Table 1. Maximum values of q, q1, and q2 for planets of the Solar System

Planet	q_{max} 10^9	$q_{1\ max}$ 10^9	$q_{2\ max}$ 10^9
Mercury	170.0	190.0	73.0
Venus	82.0	82.0	40.0
Earth	59.0	60.0	6.1
Mars	40.0	43.0	19.0
Jupiter	12.0	12.0	5.6
Saturn	6.3	6.6	3.1
Uranus	3.1	3.2	1.5
Neptune	2.0	2.0	0.98
Pluto	1.7	2.0	0.6

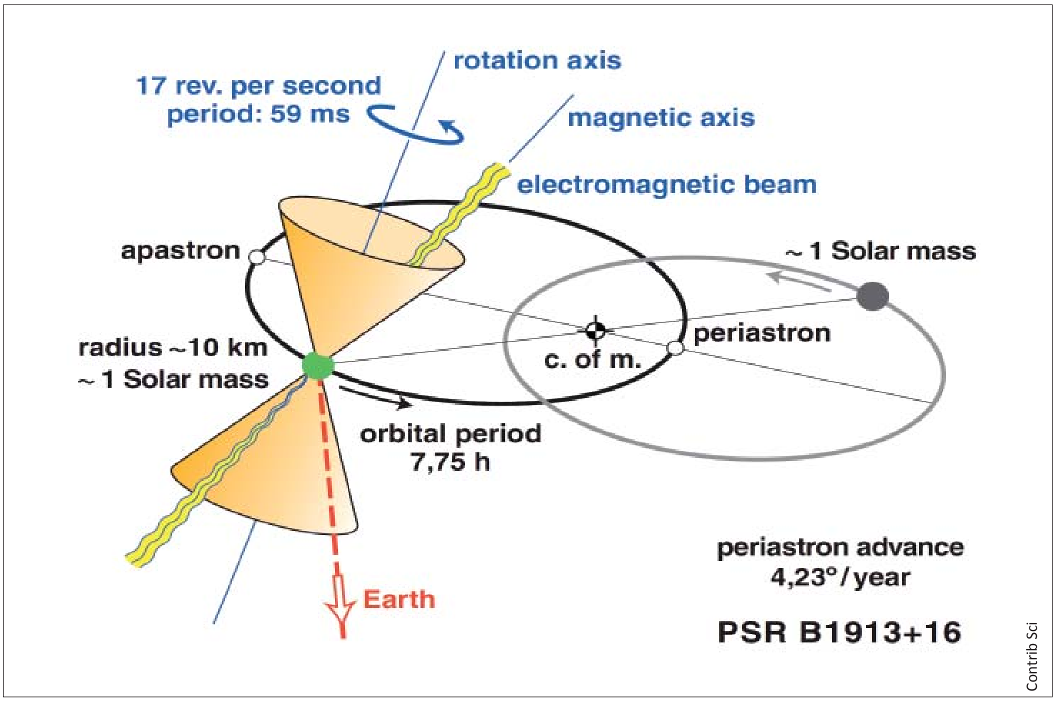

Fig. 2. Illustration of binary pulsar PSR B1913+16.

between it and its companion star. The mass of the pulsar is about one solar mass and its radius is around 10 km. Its companion star is of similar mass and radius. Their orbital period is around 8 h. Table 2 summarizes the principal parameters of the pulsar, among which the high value of the periastron advance. The last three parameters of Table 2 were obtained from the former parameters [11,25] by applying, among others, Equation (3) of the periastron advance—as applied to the two-bodies problem [Equation (11)].

The use of Equation (3) in this case is permissible because of the small value of $q \cong 3{,}6{\cdot}10^{-5}$. In this case, $q_1 \cong 1{,}8 \cdot 10^{-5}$ and $q_2 \cong 5{,}6{\cdot}10^{-6}$ are also «1, and so the Newtonian approach to Equation (3) is also permitted.

Table 2. Pulsar PSR B1913+16 parameters [10,22]

Projected semi-major axis	$a_1$sini = 2.324 ± 0.0007 light s
Eccentricity	e = 0.617155 ± 0.000007
Binary orbit period	P = 27906.98172 ± 0.00005 s
Rate of periastron advance	θ = 4.226 ± 0.002 deg yr^{-1}
Transverse Doppler and gravitation redshift	γ = 0.0047 ± 0.0007 s
Sine of inclination angle	sini = 0.81 ± 0.16
Mass of the system	M = 2.83 M_{sol} (M_{sol} = solar mass)
Pulsar mass	M_p = 1.39 ± 0.15 M_{sol}

Recently, data concerning other binary pulsars have been published [11]. Those with periastron advance > 1°/year are collected in Table 3. The so-called double pulsar (PSR J0737-3039 A and PSR J0737-3039 B), is the current best laboratory for relativistic gravitation, both for conservative effects (like the periastron advance) and dissipative effects (gravitation-wave emission). Pulsar J11411-6545, discovered in 1999, is another convenient laboratory for General Relativity due to its short orbital period (0.2 sideral days) and large eccentricity (0.17) compared to other compact binary systems made of a neutron star and a white dwarf.

In all cases, the published parameters lead to maximum values of q, q_1 and q_2 (Table 4), which are small enough, compared to unity, to allow the use of Equation (3) in both General Relativity and Newtonian mechanics with the corrected law of gravitation.

From the structure of binary pulsars one can expect that this will always be the case.

Other causes influencing the periastron advance

In previous sections, heavenly bodies were treated as particles. However their finite dimension as well as their spin-

Table 3. Other pulsars with high θ. [a]90% confidence upper companion mass limit

Pulsar	A1 sin(i) (lt $^{-1}$)	Eccentricity	θ′ (deg yr^{-1})	Binary Period (days)	Mtot (Msol)	M2 (Msol)	UprMass[a] (Msol)
J0737-3039A	1.415032	0.0877775	16.89947	0.10225156248	2.58708	1.2489	
	±1.0 10^{-06}	±9.0 10^{-07}	±6.8 10^{-04}	±5.0 10^{-11}	±1.6 10^{-04}	±7.0 10^{-04}	
J0737-3039B	1.5161	0.0877775	16.89947	0.10225156248	2.58708	1.3382	
	±1.6 10^{-03}	±9.0 10^{-07}	±6.8 10^{-04}	±5.0 10^{-11}	±1.6 10^{-04}	±7.0 10^{-04}	
J1141-6545	1.858922	0.171884	5.3096	0.1976509593	2.2892	1.02	
	±6.0 10^{-06}	±2.0 10^{-06}	±4.0 10^{-04}	±1.0 10^{-10}	±3.0 10^{-04}	±1.0 10^{-02}	
B1534+12	3.7294626	0.2736767	1.755805	0.420737299153		1.35	
	±8.0 10^{-07}	±1.0 10^{-07}	±3.0 10^{-06}	±4.0 10^{-12}		±8.0 10^{-02}	
J1756-2251	2.7564	0.180567	2.585	0.319633898			4.442
	±2.0 10^{-04}	±2.0 10^{-06}	±2.0 10^{-03}	±2.0 10^{-09}			
J1906+0746	1.420198	0.085303	7.57	0.165993045			2.867
	±2.0 10^{-06}	±2.0 10^{-06}	±3.0 10^{-02}	±8.0 10^{-09}			
B2127+11C	2.51845	0.681395	4.4644	0.33528204828			3.486
	±6.0 10^{-05}	±2.0 10^{-06}	±1.0 10^{-04}	±5.0 10^{-11}			

ning movement may influence the periastron advance. In General Relativity, Synge [24] studied the orbit of a particle in a field created by a motionless sphere with mass distribution showing spherical symmetry, and Rayner [20] extended Synge's results to the case of a central mass with uniform spinning movement.

These studies lead to terms $\dot{\theta}_1{}'$ and $\dot{\theta}_2{}''$ additive to the periastron advance $\dot{\theta}$ given by Equation (3) [23]

$$\dot{\theta}_1{}' = \dot{\theta}\left(\frac{r_0}{a}\right)^2 \frac{\left(4+e^2\right)}{10\left(1-e^2\right)} \qquad (12a)$$

$$\dot{\theta}_1{}'' = -\dot{\theta}\frac{8}{15}\frac{2^{1/2}\Omega r_0^2\cos\phi}{\left(\mu a\left(1-e^2\right)\right)^{1/2}} \qquad (12b)$$

where r_0 is the radius of the sphere, Ω its angular velocity, and ϕ the angle between the rotational axis and the direction orthogonal to the orbit plane.

For the case of the planets of the Solar System, both corrective terms can be neglected when compared to the value of $\dot{\theta}$.

For binary pulsars associated with a neutron star, the assumption of negligible diameter can be easily accepted

Table 4. Maximum values of q, q_1, q_2 [a] Maximum values of the other corrective terms that are not considered

Pulsar	q (x10^4)	q_1 (x10^5)	q_2 (x10^6)	$(v/c)^2$ (x10^6) [a]
J0737-3039A	0.27	2.61	3.04	1.01
J0737-3039B	1.27	2.61	3.48	1.16
J1141-6545	0.18	1.62	1.40	0.47
B1534+12	0.13	1.07	1.25	0.42
J1756-2251	0.52	4.76	1.18	0.39
J1906+0746	0.62	5.97	1.16	0.39
B1913+16	0.35	1.82	0.83	0.28
B2127+11C	0.94	4.09	0.90	0.30

because of the small size of this kind of star. Actually, it is known in General Relativity that the effect of the structure of the bodies becomes evident at the fifth post-Newtonian order, which makes it almost impossible to distinguish with both Solar System and pulsar observations.

Conclusions

The value of the periastron advance predicted by General Relativity in all known cases, even those regarded as natural laboratories of relativity (binary pulsars and the so called double pulsar) can also be predicted by Newtonian mechanics if a corrective term consistent with its principles is added to Newton's law of gravitation. This term can reduce to the simple form defined in Equations (7) and (8) or be any linear form of them, with coefficients ε_1 and ε_2 verifying $\varepsilon_1+\varepsilon_2 = 1$.

Newton himself was aware of the fact that formulations other than his law of gravitation would imply a perihelion shift. But during his time neither the advance of Mercury's perihelion nor binary pulsars had been detected. Thus, among the relativistic phenomena, the periastron advance is one that can be also understood in Newtonian terms by means of the addition of a corrective term to Newton's law of gravitations, consistent with Newtonian principles of mechanics.

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References

1. Bergman PG (1953) Introduction to the theory of relativity. Prentice-Hall, New York, 211
2. Brans C, Dicke RH (1961) Mach's principle and a relativistic theory of gravitation. Physical Review 124:925-935
3. Chazy J (1928) Théorie de la relativité, et la mécanique céleste. Gauthier–Villars, Paris, 1:204
4. Ref 3, 61-62
5. Ref 3, 75
6. Ref 3, Chap. III
7. Ref 3, 110
8. Einstein A (1976) La théorie de la relativité restreinte et générale. Gauthier-Villars, Paris, p. 140-141
9. Epstein R (1977) The binary pulsar–Post-Newtonian timing effects. Astrophys J 216:92-100
10. Fowler LA, Cordes JM, Taylor JH (1979) Progress report on the binary pulsar 1913+16. Austral J Phys 32:35-41
11. Hobbs G, Manchester RN, Toomey L (2014) ATNF Pulsar Catalogue, Version 1.50 [www.atnf/research/pulsar/psrcat] Updated: 6 May 2014
12. Hulse RA, Taylor JH (1975) Discovery of a pulsar in a binary system. Astrophys J Lett 195:L51-53
13. Landau L, Lifchitz E (1966) Théorie du champ. Editions MIR, Moscow, p. 271-275
14. Newton I (1687) Philosophiae Naturalis Principia Mathematica. S. Pepys, Reg. Soc. Preases, London, I:45
15. Ref 14, III:12
16. Ref 14, III:13
17. Nordtvedt Jr K (1968) Equivalence principle for massive bodies II: Theory. Phys Rev 169:1017-1025
18. Nordtvedt Jr K (1969) Equivalence principle for massive bodies including rotational energy and radiation pressure. Physical Review 180:1293-1298
19. Rainich GY (1950) Mathematics of relativity. Applied Math Series, John Wiley, New York
20. Rayner CB (1955) The effects of rotation of the central body on its planetary orbits, after the Whitehead theory of gravitation. Proc R Soc Lond A, 232:135-148
21. Rindler, W (1966) Essential relativity. Van Nostrand Reinhold, New York
22. Schild D (1961) Verifiche delle teorie gravitazionali. Rendiconti della Scuola Internazionale di Fisica "Enrico Fermi", XX Corso, C. Möller, Academic Press, New York, p. 78
23. Ref 22, 80
24. Synge JL (1952) Orbits and Rays in the Gravitational Field of a Finite Sphere according to the Theory of A. N. Whitehead. Proc R Soc Lond A, 211:303-319
25. Taylor JH, Fowler LA, Mc Culloch PM (1979) Measurements of general relativistic effects in the binary pulsar PSR 1913+16. Nature 277:437-440
26. Taylor JH, Hulse RA, Fowler LA, Gullahorn GE, Rankin JM (1976) Further observations of the binary pulsar PSR 1913+16. Astrophys J Lett 206:L53-L58
27. Taylor JH, Mc Culloch PM (1980) Evidence for the existence of gravitational radiation from measurements of the binary pulsar PSR 1913+16. Ninth Texas Symposium on Relativistic Astrophysics. Ann NY Acad Sci 336:442-446
28. Tonnelat MA (1964) Les vérifications expérimentales de la relativité générale. Masson, Paris, p 54
29. Ref 28, 11

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Resum. El problema de l'avanç del periastre, que ha basat una de les tres proves clàssiques de la teoria de la relativitat, és revisat des de les dues formulacions de la mecànica: la newtoniana i la relativitat general, i és actualitzat a la llum dels recents amidaments astronòmics en púlsars binaris. Es mostra que en la mecànica newtoniana l'addició d'un terme correctiu a la llei de gravitació de Newton, consistent amb els principis de la mecànica newtoniana, condueix a la mateixa fórmula per a l'avanç del periastre que l'emprada en relativitat general, que resulta vàlida en tots els casos astronòmics coneguts, fins i tot en el cas dels púlsars binaris tals com els PSR B1913+16 i PSR J1141-6545, i l'anomenat púlsar doble PSR J0737-3039A i PSR J0737-3039B, considerats com a laboratoris naturals de relativitat. Així doncs, entre els fenòmens relativistes, l'avanç del periastre n'és un que pot ser interpretat consistentment en termes newtonians per mitjà d'una suposició *ad hoc*.

Paraules clau: periastre · periheli · gravitació · púlsar · mecànica newtoniana