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THE HISTORY OF ALGEBRA IN ITALY IN THE 14TH AND 15TH CENTURIES. SOME REMARKS ON RECENT HISTORIOGRAPHY

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Summary: *The solution of the algebraic equations of third and fourth degree by Italian algebraists in the first half of the 16th century is considered the beginning of the development of modern mathematics. These results have their roots in Italian vernacular algebra that was taught in Abbacus School and written in a chapter of abbacus treatises since the beginning of the 14th century. In recent years many of these treatises has been published and studied. Particular attention has been paid to the origins of Italian vernacular algebra that does not seem linked to al-Khwarizmi's and Fibonacci's tradition. In this paper we make a survey of the main 14th century treatises and give some contribution to the problem of the origins.*

Key words: *History of algebra, Italy, 14th and 15th century*

1. Introduction

The solution of the algebraic equations of third and fourth degree by Italian algebraists in the first half of the 16th century is considered the beginning of the development of modern mathematics in Europe. These results have their roots in Italian abbacus schools where algebra was studied and improved from the 14th century.

The first attempt to describe the development of algebra in Italy in 14th and 15th centuries was made in: Pietro Cossali, *Origine trasporto in*

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Italia, primi progressi in essa dell'Algebra (Cossali, 1797-99). In the first volume, Cossali illustrates the algebra chapter in Leonardo Pisano's *Liber abaci*, which he correctly dates at 1202. Until then, in fact, historians of mathematics, including Montuclà, maintained that Leonardo's treatise was written in the fourteenth century. Before Cossali, the sources examined for charting the history of algebra in Italy were only printed texts. Thus, the first document examined by historians was the algebra chapter contained in Luca Pacioli, *Summa de Arithmetica Geometria Proportioni proportionalità* (Pacioli, 1494).

The name Leonardo Pisano or Fibonacci was already known, for Pacioli attributed to him much of what he wrote. However, Cossali was the first to read the algebra chapter of the *Liber abaci* and to compare it with Pacioli's treatment, by which he arrived at the hypothesis that there were probably other algebra treatises between Pisano and Pacioli, although he was unable to find them. Nevertheless, he came to the conclusion that algebra was imported into Europe from Arabic countries by Fibonacci, after which it spread from Pisa into Tuscany, mainly in Florence, then into other Italian countries and later into other European countries. This conclusion, only partially correct in the light of the richer documentation we have today, was accepted for about two centuries and half. In particular, it is necessary to point out that Cossali completely ignored the Latin translations of Al-Khwarizmi's *Al-jabr,* even though he correctly recognized that algebra had its origins in this treatise (Franci, 1989).

A second important step in the historiography of algebra in the 14^{th} and 15^{th} centuries was made by Guglielmo Libri in his *Histoire des Sciences Mathematiques en Italie* (Libri, 1838-1841). Libri's reconstruction is based on many manuscripts he had seen in the libraries of Paris and Florence and some of his own property. In the first volume Libri recognizes that algebra entered into Europe by means of the Latin translations of Al-Khwarizmi's *Al-jabr* and publishes that of Gerardo da Cremona. In the second volume, he analyzes Fibonacci's contributions and publishes the whole fifteenth chapter of *Liber abaci,* which is devoted to algebra. For the period immediately after Fibonacci, Libri affirms there were no followers in the thirteenth century. On the other hand, he quotes some abbacists of the fourteenth century such as Paolo dell'Abaco, Paolo Gerardi and Giovanni Danti, whose treatises contain algebra. In the third volume, Libri illustrates Luca Pacioli's contributions and mentions having seen some anonymous treatises in which third and higher degree equations are solved, even though the solutions are wrong. In fact, he says that in these treatises the third degree equations are solved with formulas similar to those for the second degree equations, while for some other equations "des règles bizarres fondeés sur de faux principles" are given. He further publishes two large excerpts from manuscripts of his own property where these alleged solutions are illustrated[1] .

1. These excerpts contain the wrong rules of Gerardi and the non numbered rules of Dardi, which we illustrate in a paragraph below.

The next important contribution to the medieval Italian historiography of algebra was made by Baldassarre Boncompagni, who not only published all the treatises of Leonardo Fibonacci, but also illustrated many Italian manuscripts of the 14th and 15th centuries containing important chapters on algebra, although without going into details (Boncompagni, 1854; 1857-1862).

It was only in the 1970s, in accordance with Boncompagni's indications, that Gino Arrighi (1906-2001) began to study and to transcribe many important abbacus treatises, but his focus was general and not only for algebra.

A virtually complete list of manuscripts containing algebra has been possible only from nineteen-eighty, when Warren Van Egmond published a catalogue of Italian abbacus manuscripts (Van Egmond, 1980).

This was the situation when, about thirty years ago, Laura Toti Rigatelli and I began to be interested in the history of the Italian medieval algebra. With the help of Van Egmond's catalogue, we made a list of abbacus manuscripts containing a chapter on algebra, and with our students we began to transcribe and to study many algebra treatises of the 14th and 15th centuries. Some of these texts have been published in the series *Quaderni del Centro Studi di Matematica Medioevale dell'Università di Siena*. We subsequently wrote two surveys on the most interesting things we had found in these manuscripts, paying more attention to the new achievements as regards the *Liber abaci* (Franci & Toti Rigatelli, 1985; 1988).

Some years ago I again studied the fourteenth century algebra treatises, pointing out the elements that differ from Al-Khwarizmi and Fibonacci (Franci, 2002) in order to make the search for other possible sources easier.

The first algebra treatises to circulate in Italy were probably Gerardo's Latin translation of al-Khwarizmi's *Al-jabr* and the third part of the fifteenth chapter of *Liber abaci.* The first Italian vernacular algebra treatises, however, are very different, so the question of finding their sources arises. Before tackling this problem, it is opportune to give a glance to the environment in which algebra developed.

The algebra treatises of Italian Middle Ages are usually a chapter of a larger text named *abbacus treatise* (trattato d'abaco); algebra was in fact taught in *abbacus schools* (scuole d'abaco) where people learned practical mathematics in order to employ it in business. The teachers in these schools, known as *abbacus masters* (maestri d'abaco), often wrote treatises which, although having many subjects in common with Leonardo's *Liber abaci,* are different from it in many respects (Franci, 2003a).

Only a part (about a third) of the surviving abbacus treatises contained a chapter devoted to algebra. Algebra in fact was not strictly necessary to the mathematical education of merchants. Very few are the treatises entirely devoted to it. The majority of abbacus treatises consist of a collection of solved problems, sometimes preceded by a rule, only few having a longer theoretical part. The treatment of algebra they contain obviously follows this style. The number of pages and the topics covered in the algebra chapters are variable, the typical contents

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bring: calculations with radicals, calculations with monomials and polynomials, rules for solving algebraic equations, problems solved by algebra. Not all the subjects are always present; more often than not we have only a collection of solved problems preceded by a rule.

The main sources for the history of fourteenth-century Italian algebra are fifteen manuscripts belonging to the 14th century and three to the 15th century[2]. In recent decades, many of the algebra treatises contained in these manuscripts have been transcribed and studied, providing us with a better knowledge of the algebra of that period. Nevertheless, some important historical problems still remain to be solved, the first one being that of the sources.

2. The beginning of Italian vernacular algebra

We do not know of any Italian algebra treatises written before the fourteenth century; the oldest actually date from the first decades of the 14th century and are as follows:

2.1. *Paolo Gerardi's* Libro di ragioni

In 1978 Warren Van Egmond published what he believed to be "the earliest vernacular treatment of algebra"; it refers to a chapter from the *Libro di ragioni* by Paolo Gerardi, written in Montpellier in 1328[3] and contained in the manuscript Magl. Cl. XI, 87 at the Biblioteca Nazionale of Florence (Van Egmond, 1978).

The chapter on algebra is the last one in the treatise (ff. 63r-70r). Gerardi's treatment contains fifteen rules. Each rule, entitled in red "*regolla della chosa*" or "*delle chose*" or "*choza*", gives the algorithm to solve an equation. The rules concern the following equations[4]:

1. $ax=b$	2. $ax^2=b$	3. $ax^2=bx$	
4. $ax^2+bx= c$	5. $bx=ax^2+c$	6. $ax^2=bx+c$	
7. $ax^3=b$	8. $ax^3=\sqrt{b}$	9. $ax^3=bx$	10. $ax^3=bx^2$
11. $ax^3=bx^2+cx$			
12. $ax^3=bx+c$ (!)	13. $ax^3=bx^2+c$ (!)	14. $ax^3=bx^2+cx+d$ (!)	
15. $ax^3+bx^2=cx$			

The equations marked by (!) are accompanied by a wrong rule: In fact, they are solved as they if were $ax^2=bx+c$ in the first two cases and as $ax^2=bx+(c+d)$ in the last case. In what follows we refer to these rules as Gerardi's wrong rules. The other equations are correctly solved in an obvious way. Gerardi begins his treatment of algebra directly with the rules,

2. A complete list is given in the Appendix.

3. This information is found at the beginning of the treatise where we read " .. le regole e il corso dell'ambaco facte per Paolo Gerardi di Firenze , …, anno domini 1327 a dì 30 di gennaio secondo lo corso di Mompeslieri." Recall that in those times the year began on 25th March, thus according to our calendar the treatise was written in 1328.

4. Here and in the sequel we write the equations using the modern symbolism.

without explaining the meaning of the terms *cosa*, *censo*, *cubo,* which he uses to represent the unknown and its powers. Each rule is followed by a problem, and simple rules for algebraic calculations are illustrated within the solution to the problems.

The first six rules are the same as those presented by al-Khwarizmi and Fibonacci, but Gerardi's treatment of algebra is quite different[5]. Not only the geometrical proofs are missing, but so is the introduction of algebraic quantities and rules of calculations, and most parts of the problems are different. In particular, the fourth problem concerns a calculation of interest, while the fifth deals with business trips.

2.2. *Jacopo de Florentia's* Tractatus algorismi

In 2000 Jens Høyrup published what he claimed to be "the earliest vernacular algebra so far known" (Høyrup, 2000, 2001, 2006). He affirms that the text he presents is a chapter from Jacopo de Florentia's *Tractatus algorismi*, written in Montpellier in 1307. Actually we know of three copies of this treatise, one of the earliest Italian vernacular abbacus. They are contained in the manuscripts: Ricc. 2236, Biblioteca Riccardiana (Firenze); Ms 90, Biblioteca Trivulziana (Milano); Vat. Lat. 4826, Biblioteca Apostolica Vaticana. The first one can be dated from the beginning of the 14th century, the second and third from the beginning and the mid of the 15th century, respectively. All of them begin:

> Incipit tractatus algorismi compilatus a magistro Jacobo de Florentia apud Monte Pesulanum anno domini millesimo trecentesimo septimo in mense septembrio.

The chapter on algebra is contained in the last manuscript only. This suggests that this chapter on algebra might be a later interpolation, although from a comparative analysis of the style of the various chapters, Høyrup concludes that algebra has a common authorship with the other sections of the treatise, and so believes Jacopo's algebra to be the earliest vernacular Italian algebra so far known.

Algebra occupies folios 36v-43r and is preceded by a series of geometrical problems, then followed by some problems solved by arithmetical tools. Jacopo begins his treatment with the six classical rules listed in the same order as Gerardi and followed by one or more problems. After the 5th rule, he remarks:

> But keep in mind that all the computations leading back to this rule can be answered with two answers, but only some of them, and there are some for which you ought to take one answer, and some the other.

Then he solves three problems, one for each case. After the 6th rule, he writes:

5. We also remark that al-Khwarizmi and Fibonacci list the first three rules in a different order, that is, 3, 2, 1.

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> Here I end the six rules combined with various examples, and begin the other rules that follow the six mentioned above, as you will see.

The other rules concern the following equations and are not followed by problems.

7. $ax^3=b$	8. $ax^3=bx$	9. $ax^3=bx^2$	
10. $ax^3+bx^2=cx$	11. $bx^2=ax^3+cx$	12. $ax^3=bx^2+cx$	
13. $ax^4=b$	14. $ax^4=bx$	15. $ax^4=bx^2$	16. $ax^4=bx^3$
17. $ax^4+bx^3=cx^2$	18. $bx^3=ax^4+cx^2$	19. $ax^4=bx^3+cx^2$	
20. $ax^4+bx^2=c$			

Like Gerardi, Jacopo does not previously introduce the terms of algebra he uses, which are *cosa, censo*, *cubo* and *censo di censo*. He also teaches algebraical calculations during the solution of problems. Jacopo's treatment resembles that of Gerardi, not only in the organization of the matter but also in the coincidence of many rules and problems. In particular, the problems relative to the rules 1,3,4,5 are different only in the numerical data. Jacopo, however, does not present the wrong rules.

At this point a problem naturally arises: what are the sources of Jacopo's and Gerardi's algebras? Their treatment, in fact, follows a pattern very different from that of al-Khwarizmi and Fibonacci, not only because they both omit the geometrical proofs of the rules and present equations of degree higher than two, but also because they apply algebra to solve different kinds of problems.

Høyrup affirms that the above-mentioned features of Jacopo's and Gerardi's algebras are present in some Arabic texts. Nevertheless, the lack of any Arabism in their texts excludes a direct derivation from Arabic sources. Thus, Høyrup develops a hypothesis of the influence of a Romance speaking area, which in turn would have been influenced by Arabic sources. The presence of the wrong rules in only one of the texts, however, suggests that Jacopo and Gerardi might have drawn from different sources, even if they both wrote their treatises in Montpellier. Even though we are not sure whether Jacopo was the first to present the twenty rules listed above, in what follows we refer to them as Jacopo rules.

In 2007 Høyrup edited the entire Jacopo treatise (Høyrup, 2007). In his long introduction, he reaffirms that Jacopo's algebra is the earliest Italian vernacular algebra and repeats his hypothesis on its origins. In a review of Høyrup's book, Jeff Oaks presents many arguments against Høyrup's thesis that algebra in Vat. Lat. 4826 was already contained in Jacopo's original treatise written in 1307 (Oaks, 2009). Oaks also contests Høyrup's related thesis that Jacopo learned algebra not from the environment of Montpellier or Italy but from an unknown "area?" located in or encompassing the Catalan region, which is not well proved. Before entering into the details of this question, it is necessary to illustrate the others surviving algebra compilations that are almost contemporary to those of Gerardi and Jacopo.

2.3. *Tractato dell'arismetricha (Ricc. 2252)*

The manuscript Ricc. 2252, Biblioteca Riccardiana of Florence, in the folios 1r-71v contains a *Tractato dell'arismetricha* written in Pisa that by internal evidence can be dated at about 1320[6]. After a first part, devoted to teaching the operations with natural numbers, the author begins to explain the rules for solving the problems, and after the rule of the three and that of double false position he introduces "the rules of the greater thing with some examples by means of which all the problems are solved or can be solved"[7]. The sixteen rules, numbered in red ink, coincide with the first sixteen of Jacopo, but they are presented in a different order. After the first six, all the binomial of third and fourth degree are listed. Unlike the other texts we have examined, the first five rules are illustrated by numerical examples. Those concerning the 4th and 5th rule are: $x^2+10x=39$ and $x^2+21=10x$, which coincides with those we find in al-Khwarizmi. Furthermore, while in the rules the author speaks of "censi", "cose" and "numeri", in the examples the numbers are named "dragme", which recall al-Khwarizmi's "dirhams". Rules 6, 7, 8, 9, 10, 16 are illustrated by a problem, which for rules 8, 9, 10 is the same as that by Gerardi. Ten problems solved by algebra follow, and before each problem the author again writes in red ink the rule to which it refers; the rules are 1-2-3-4-6 on his list. These problems are different from those in Jacopo and Gerardi. In conclusion, it seems the author of this text has drawn his algebra from sources different from those of Jacopo and Gerardi.

2.4. *The manuscript 1754 of Biblioteca Statale of Lucca*

Manuscript 1754, Biblioteca Statale of Lucca[8], dated by internal evidence at about 1330, contains an abbacus treatise in which, after a well organized treatment of the commercial questions, we find a chapter devoted to algebra

> The third book containing the rules of the thing with examples, with other good problems that I have written by number and by root, as you will find[9].

Sixteen rules follow, each preceded by the inscription "Reghola della chosa". The rules coincide with the first sixteen of Jacopo, but they are presented in the same order as Ricc. 2252. Only the first six rules are illustrated by a problem. These examples are similar to those presented by Jacopo and Gherardi; sometimes they also coincide in the numerical data. At the end of the rules we find another five problems solved by algebra.

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6. This interesting manuscript has yet to be studied by anybody.
7. "Regole della cosa maggiore con certi esempi per li quali si fanno o si possono fare tutte le ragione".
8. The whole manuscript is edited in: Scuola Lucchese, 1973.
9. "Terso libro e' serae le regole della chosa con asempri, chon altre buone ragioni di vantaggio; le quali i' òe scritti per numero e per radicie, sì chome voi troverete".

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In the last part of the manuscript there is another chapter on algebra that begins:

> Here we will begin the rules of *aligibra amichabile,* that is, of the thing as they are expressed in the vernacular, by which all the abbacus problems are solved and also many of geometry[10].

Next the six classical rules follow, each of them illustrated by a problem that differs from that of the first part only in so far as numerical data. After the six rules we read the following remark:

> These are 6 rules of *aligibra amichabile*, and by these six ways almost all the problems are solved, by which one are able to bring back his problem to these rules. But there are other rules of *chubi* and *censi* that are seldom needed because each problem can be brought back to one of those six ways. Nevertheless, I will propose them to you without examples so that, if you need them, you shall know them[11].

The rules that follow also coincide in the order with the 7-13 of Jacopo and are not followed by a problem.

The second version of algebra, while very similar to the first one, does not seem to be drawn from the same source. Elements in favour of this claim are the different order in the list of rules, the different numerical data in the problems, and finally the different name given to the discipline: "regola della cosa" in the first case and "aligibra amichabile" in the second case. We remark that this is the first occurrence of the name "aligibra amichabile" in the vernacular tradition.

2.5. *Some conclusions*

Analyses of the algebra treatises we have made above enable us to tackle two problems concerning the origins of Italian vernacular algebra:

(1) What is the first Italian algebra treatise?

(2) What are its sources?

As regards problem (1), we recall that there are two candidates: Gerardi in Van Egmond's opinion and Jacopo in Høyrup's opinion. Høyrup's claim is based on the belief that the al-

10. " Quie inchomincieremo le regole dell'aligibra amicabile, ciò della chosa sono dette volgarmente; per le quali tutte le questioni dell'anbacho se n'asolveno e ancho molte di giemetria".

11. "Queste sono 6 reghole dell'aligibra amicabile, e per questi 6 modi quaçi tutte le ragioni s'asolvono a chi sa indugiare la sua ragione a questi modi. Ma sono anchora altre regole di chubi e de' ciensi le quali rade volte biçogniano in però che ogni ragione si puote indugiare ad alcuno di questi 6 modi ma nondimeno io te le porrò quelle sensa gli asempri acciò se pure ti biçognassero, che tu le sappi".

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gebra chapter in the 15th century copy of Jacopo's treatise was already present in the original. As I stated above, Oaks contests this assertion and I fully share his arguments. I further believe that not even Gerardi's treatise may be the first one. Indeed, in my opinion the solution to problem (1) is linked to the problem of dating manuscripts. For the most part, they are dated by the watermark on the paper or by internal references to some dates contained in astronomical or commercial problems. It is obvious that such dating has a range of error. Now if we look at the manuscripts Ricc. 2252 and Lucca 1754 and take into account the range of error in their dating, we may conclude that they are almost contemporary or perhaps earlier than Gerardi's treatise. So we are not able to say which the first is. Furthermore, they seem to be rather independent one from another. This conclusion is also important in providing an answer to problem (2).

Although independent each from one another, the treatises examined above share some common features that revert back to common sources that cannot be identified with Latin translations of al-Kwarizmi or Fibonacci's *Liber abaci*. It seems likely that they ultimately derive from Arabic sources; nevertheless, they are different from any Arabic treatise so far investigated by historians of mathematics. Since one of the characteristic of early Italian algebra is its extensive use in solving commercial problems, Høyrup develops the hypothesis that its source may ultimately be found in *muhalameth* mathematics (Høyrup, 2007:159). Unfortunately, this field of Arabic mathematic is still little studied, so no text containing an algebra treatise similar to the Italian ones is available!

As regards the problem of how this algebra reached Italian abbacists, we have already mentioned how Høyrup arrived at the hypothesis of the influence of a Romance speaking area, which in its turn would have been influenced by Arabic sources, and how he further located this area in the Catalan region. He bases his hypothesis on the conviction that Jacopo's and Gerardo's algebras are the earliest in Italian, and therefore form the basis of the whole subsequent Italian production. If we take into consideration the analysis of the algebra treatises made above, and what we have said about the dating of the manuscripts, it is evident that Høyrup's premise is false.

In the light of what we currently know, I believe that we can only conclude that in the first decades of the 14th century a knowledge of algebra independent from al-Khwarizmi and Fibonacci was widespread in Tuscany, and that its main characteristics are the solution of equations of degree greater than two as well as the application to solve commercial problems, whose origins are probably Arabic. However, we are absolutely unaware of the way in which this algebra reached Italy.

3. Dardi's *Aliabraa-Argibra*

The problem of the sources for Italian algebra becomes even more complicated if we look at the first Italian vernacular treatise entirely devoted to this subject. This is entitled *Aliabraa-Argibra* and was written in 1344 by Master Dardi of Pisa. So far five copies of *Aliabraa* are known. They are contained in the manuscripts: Chigi M.VIII.70 (c.1395), Biblioteca Apostolica Vaticana; I.VII.17 (c.1470), Biblioteca Comunale of Siena; Ash 1199 (c.1495), Biblioteca Mediceo-Laurenziana of Florence, Heb.1022 (1473, d), Bibliothèque Nationale of Paris; Arizona State Library of Tempe (USA) (sec. XV). The first three copies are anonymous, while the last copy gives us the name of the author and the date of compilation of the treatise. Furthermore, it was the point of departure for the translation into Hebrew contained in the Paris manuscript. The translator was Mordechai Finzi of Mantova, who is also well known for his translation into Hebrew of Abu Kamil's *Algebra*. The three Italian copies are almost identical; the first was written in a vernacular of Northern Italy, while the other two are in a Tuscan vernacular (Van Egmond, 1983; Maestro Dardi, 2001).

In the introduction, the author explains that "the title of the book is called Aliabraa in Arabic, which in the vernacular means uncovering of a subtle matter". Next he shortly presents the content of the treatise and defines the terms *cosa, censo, cubo, censo di censo*. After a chapter devoted to calculations with radicals, the algebra treatise begins. It is divided into two parts.

In the first part the classical six equations of first and second degree are presented, together with geometrical proofs and some simple rules for calculating with the unknown and its powers. This part resembles al-Khwarizmi. The second part is on the other hand highly original. It contains 198 rules, 194 of which are numbered, while the other four are not. Each rule is followed by one or more problems.

We do not include the list here, but confine ourselves to remarking that the equations are obtained by combining in different ways the following terms: a, ax, ax^2, ax^3, ax^4,$\sqrt{a}$, $\sqrt{ax}$, $\sqrt{ax^2}$, $\sqrt{ax^3}$, $\sqrt{ax^4}$, $\sqrt[3]{a}$, $\sqrt[3]{ax}$, $\sqrt[3]{ax^2}$, $\sqrt[3]{ax^3}$, $\sqrt[3]{ax^4}$, where "a" is a positive integer. In the main, Dardi limits himself to cases with only two or three terms, and only in seventeen cases does he treat equations with four or more terms. It seems important to remark the presence of equations like $ax^4+bx^2=c$ and $ax^6+bx^3=c$. Almost all the proposed equations are reduced to one of the six first cases or to a binomial equation like $ax^n=b$, $2\leq n\leq 12$, either by substitution or by rising to an appropriate power. The sample problems for the numbered rules are all one of the following types: *Divide 10 into two parts such that..., Find one (or two or three or four) numbers such that...* The terms of the problems are always such that the equations are obtained quickly.

Between chapters 182 and 183, four non-numbered rules are inserted, and this is perhaps one of the most interesting parts of the treatise. The author warns the reader that, unlike the others, these rules are not as generally valid, since as he says the rules presented are valid only for equations which come from problems like those proposed. In these cases, as well as in the others, Dardi writes down the rule without any explanation about the way it is obtained. The equations in question and relative solving rules are

(i) $ax^3+bx^2+cx=d$ $\qquad x=\sqrt[3]{(c/b)^3+d/a} - c/b$

(ii) $ax^4+bx^3+cx^2+dx=e$ $\qquad x=\sqrt[4]{(d/b)^4+e/a} - \sqrt{d/b}$

(iii) $ax^4+dx=bx^3+cx^2+e$ $$x=\sqrt[4]{\left(\left(\frac{c}{2a}\right)^2+\frac{e}{a}\right)}+\frac{b}{4a}-\sqrt{\frac{d}{2b}}$$

(iv) $ax^4+cx^2+dx=bx^3+e$ $$x=\sqrt[4]{\left(\left(\frac{c}{2a}\right)^2+\frac{e}{a}\right)}+\frac{b}{4a}-\sqrt{\frac{d}{2b}}$$

The problems illustrating rules (i) and (ii) are relative to a calculation of interest, the others ask us to divide 10 into two parts, x, 10-x, such that

$(x(10-x))/(2x-10)=\sqrt{N}$, with N a given natural number[12].

While the first part of *Aliabraa* is very close to al-Khwarizmi's text, borrowing its examples and proofs, nothing is known about possible sources for the second part.

4. The algebra treatises of the second part of the 14th century

The algebra treatise contained in codex Ricc. 2263 of Biblioteca Riccardiana of Florence, dated by watermark about 1365 (Anonimo, 1994), is very interesting, because here we find for the first time Gerardi's and Jacopo's rules listed together. This text, entitled *Trattato dell'alcibra amuchabile*, begins with the rules of signs and rules to operate with radicals. The next twenty-four equations follow. The part concerning the first six rules resembles Jacopo's word for word. Rules 7 to 13 coincide, although in a different order, with that so numbered in Gerardi, and they are also followed by the same problems. Rules 14 to 24 coincide with Jacopo's last ten rules and are not followed by a problem. In conclusion, this set of rules is almost the exact union of Jacopo and Gerardi. In fact, only Gerardi's third wrong rule and Jacopo's 10^{th} rule are missing. Forty problems solved by algebra subsequently follow, most of them of business type. It is interesting to point out that in some of these problems the rules for operating with the algebraic fractions are illustrated (Franci, 1992: 325-334).

The other Italian surviving algebra treatises that we know from the second part of the 14 century date back to the last decade of the century. Many of them share a common feature; they present a greater number of equations than Gerardi and Jacopo. Many of them like Fond. Prin. II. III. 198, 2Qq E13, Ricc. 2252 (ff 160r-169r), Pal. 312 (Anonimo, 1998), include Gerardi's wrong rules and Dardi's non-numbered rules followed by the same problems. The authors of these treatises appear to have copied from other texts without criticism. They seem to be completely unaware that some rules were wrong and others valid only for a limited class of problems.

12. Guglielmo Libri owned what is now the Ashburnham manuscript, and at the end of note XXXI in the third volume of his "Histoire…" he prints the section concerning the four special equations. For a good analysis of these equations see Van der Waerden, 1985: 47-52.

Very different is the quality of the chapter on algebra, included in the manuscript Fond. Prin. II.V.152 of Biblioteca Nazionale of Florence, written by an anonymous Florentine master, in the last ten years of the fourteenth century. Here in fact we find an interesting improvement on the solution of the third degree equations. Algebra occupies folios 145r to 180v (Anonimo, 1988). After a very detailed exposition of calculations with polynomials, twenty-two rules follow that coincide with the twenty by Jacopo plus $ax^4+c=bx^2$, $ax^4=bx^2+c$, which are the logical companions of the last rule in Jacopo's list. Each rule is explained by two or three problems. At the end of this part we read the following remark

> Here we explain the 22 rules of algebra with some examples. Now I will continue with some rules besides these, which are fine. And from this you will understand as many rules you could make on these equations, although it is difficult to make them[13].

The rules are relative to the following equations:

23. $ax^3+bx^2=c$, 24. $ax^3= bx^2+c$, 25. $ax^3+c=bx^2$.

The author claims that the solution to 23 is, in modern notation, $x=y-(b/3a)$, where y is such that $y^3=3(b/3a)^2y-(c/a+2(p/3a)^3)$. In the examples he solves, the last equation is solved by trial and error. The rules to solve the equations 24 and 25 are similar except for obvious change of signs. In each case the rule is followed by a numerical example and by a problem. We remark that what the anonymous author does is to turn an equation like $x^3+px^2+q=0$ into $y^3+p'y+q'=0$ by the substitution $y=-p/3$. However, he lacks a formula to solve the latter. Rather he solves it by trial and error in any particular case (Franci, 1985).

It is very interesting to remark that the author of the treatise claims to be the inventor of the rules to solve the equations 23-24-25.

> Since we have proved these three rules besides the twenty two, it is possible that many rules exist, but so far we have made no more, so I will leave some space so that if at another time we wish to make more rules we can write them here[14].

The first and third of the afore-mentioned rules are also presented in the almost contemporary manuscript Conv. Sopp. G.7.1137 (ff. 150v-151v), Biblioteca Nazionale of Florence. The problems that illustrate these rules are different from those of the previous text.

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13. Qui abbiamo chonpiuto di dichiarare le 22 reghole del'algibra, chon alcuni asenpri, ora seguiterò alcuna regola fuori di queste che sono belle e per quelle chonprenderai chome molte regole si potrebono fare sopra a questi uguagliamenti ma sono d'asai sentimento a volerle fare.

14. "Sichome abiamo dimostrate queste tre regole oltre alle 22, è posibile che asai regole siano, onde non abiendo per ora fatte più lascerò di spazio per insino a la metà del quaderno a ciò che, se altra volta più reghole facessimo si possano seghuire." <f. 166r>.

Until now we have not found these rules in any other treatise. Nevertheless, we must remark that in the 15th chapter of *Ars Magna* (1545), in order to solve the equation $ax^3+bx^2=c$, Girolamo Cardano transforms it exactly in the same way as the anonymous abbacists of the 14th century. The great difference is that Cardano knew how to solve the transformed equation.

From the analysis made until now it might seem that the 14th century abbacists were completely unaware of the treatment of algebra by al-Khwarizmi and Fibonacci. This is not so, however. In fact, in the manuscript MA 334 (c. 1400), Biblioteca Angelo May of Bergamo, at ff. 63r-66r, we indeed find a translation into the Venetian vernacular of the section of *Liber abaci* containing the geometrical proofs of the second degree compound rules, followed by a treatment of algebra that follows the pattern we have seen so far. In this section, we find twenty five rules each illustrated by a problem. The first eighteen rules coincide with those of Ricc. 2263. Next there are the four non-numbered rules by Dardi, followed by $ax4+bx^2=c$, $ax^4+c=bx^2$, $ax^4=bx^2+c$. Gerardi's wrong rules are not included.

An identical treatment of algebra, including the translation of the geometrical proofs from *Liber abaci*, is contained in ff. 13r-30v of Magl. XI. 120 (c. 1400), Biblioteca Nazionale of Florence, which is written in a Tuscan vernacular.

The 14th century abbacists' knowledge of the Latin origins of algebra is also evidenced by a translation into the vernacular of Gerardo da Cremona's Latin translation of al-Khwarizmi's *Al-jabr,* contained in ff. 84r-107v of the manuscript Fond. Prin. II. III. 198, Biblioteca Nazionale of Florence, while the manuscript Urb. lat. 291 (c.1400), Biblioteca Apostolica Vaticana, contains a vernacular translation of another Latin translation of *Al-jabr* (Franci, 2003b) and a translation of the third part of the fifteenth chapter of *Liber abaci*.

I believe that the study of Urb. Lat 291, until now ignored by the historians of medieval algebra, may be of great interest. It seems in fact to be a compendium of the whole 14th-century Italian vernacular algebra. It presents a translation of the 14th chapter of *Liber abaci* (devoted to radicals), a translation into Italian of a Latin edition of the *Al-jabr*, a translation of the third part of the 15th chapter of *Liber abaci* followed by other problems solved by algebra, a list of thirty five rules containing all the rules of Jacopo and Gerardi, and the four non-numbered rules of Dardi followed by another set of problems solved by algebra.

5. Italian vernacular algebra of the fifteenth century

In the 15th century, algebra continues to be presented in appropriate chapters of abbacus treatises, and their contents are often copied word by word from texts of the previous century. Many 15th century treatises continue to present the wrong rules of Gerardi, which proves that their authors copied without criticism[15]. A very different matter are the algebra chapters con-

15. In this regard, we remark that the first treatise by a Portuguese author that has come down to us containing a treatment of algebra, namely Bento Fernandes's *Tratado da arte de arismetrica* (1555), contains a chapter on algebra quite similar to that of the above quoted manuscript Pal 312 including Gherardi's wrong rules. See: Silva, M.C., 2008.

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tained in some large treatises that may be considered as encyclopedias of the mathematics taught in the abbacus schools, they are entitled *Praticha d'arismethrica* and are contained in the manuscripts L.IV. 21, Pal. 573, Ott. lat. 3307[16]. The first one was written by Benedetto da Firenze, the others by an anonymous Florentine master who claims to be a student of a certain Agostino Vaiaio. The long chapters devoted to algebra in these treatises are very similar[17]. They are divided into many sections. The first one has a general and introductory character and begins with a translation into the vernacular of the first part of the Latin version of *Al-jabr* by Gerardo da Cremona. A list of rules for solving third and fourth degree equations follows, all the equations being binomial or trinomial equivalent to a second degree equation. There are no traces of Gerardi's wrong rules or Dardi's non-numbered rules. This section ends with the solution of many simple problems. Each of the following parts is devoted to the presentation of a series of problems drawn from treatises of other authors, for which biographical notes are also given. Leonardo Pisano, Antonio de' Mazzinghi (sec. XIV) and Giovanni di Bartolo (sec. XV) are present in all the treatises, while M° Biagio (sec. XIV) appears only in L.IV.21 and M° Luca (sec. XV) and Agostino Vaiaio (sec. XV) only in the other two texts.

These treatments of algebra not only show that their authors had a sound historical knowledge of the subject, but also that they exerted a criticism. They in fact deliberately omitted to write out the wrong rules for the third degree equations, as emerges from the following remark by M° Benedetto:

> We might write certain equations that were given for the third powers in ancient times, but these rules are very obscure and more doubts are in the solutions than in the enunciations, so those given are sufficient[18].

In spite of M° Benedetto's warning, many abbacists of the 15th century include the wrong rules in their texts. Among them we may recall Raffaello Canacci, who has left to us two treatises entirely devoted to algebra contained in the manuscripts Ricc. 2265 (c.1490), Biblioteca Riccardiana of Florence, and Pal. 567 (c. 1495), Biblioteca Nazionale of Florence[19]. Canacci's treatment includes the geometrical proofs for the second degree equations and calculations with radicals, monomials and polynomials. His list of sixty five rules includes Gerardi's wrong rules and Dardi's four non-numbered rules, illustrated by the same problems we find in the texts of the ancient authors.

16. For a good description of the content of these treatises see: Arrighi, 1965; 1967; 1968. The three papers are also reprinted in Arrighi, 2004.

17. The algebra chapter in L.IV.21 has been completely edited in the following publications: M° Benedetto da Firenze, 1982; M° Biagio, 1983; M° Antonio de' Mazzinghi, 1967; Giovanni di Bartolo, 1982; Leonardo Pisano, 1984.

18. "… certe aguagliationi le quali anticamente sopra e chubi davano si potrebbero scrivere , ma sono regole molto offuscate et in più dubi si risulta nella solutione che nel proporro del chaso, queste date adunque bastino, …" <L.IV.21, f. 430v.>

19. The last manuscript has been completely edited in: Procissi, 1954 and Canacci, 1983.

The manuscript Ital 578, Biblioteca Estense of Modena, contains the only original treatment of algebra in Italy from the 15th century (Anonimo (sec.XV), 1986). Algebra occupies the folios from 5r to 18v. The treatment begins with a list of the powers of the unknown, which range from simple numbers to an unknown of ninth degree, which are denoted by the following abbreviations : N (number), C (x), Z (x^2), Q (x^3), ZZ (x^4), C di ZZ (x^5), Z di Q (x^6), C di Z di Q (x^7), Zdi ZZ (x^8), Q di Q (x^9), where C, Z, Q are the first letters of 'Cossa', 'Zenso'and 'Qubo', which are the names of the first three powers of the unknown. The author associates one 'grado' (degree) to each term, beginning with 0 for N, 1 for C, 2 for Z, and so on. The list of the powers is followed by an explanation of how to perform the four basic operations with them. Here we find the first original contribution. In fact, as regards multiplication and division, the author suggests that the result can be obtained by adding or subtracting the degrees. The author's originality also continues to reveal itself in listing the equations; rather than giving a separate rule for every equation; he divides his equations in eighteen basic types and gives one general rule that can be used to solve any equation of that type. Thus, $ax^2+bx=c$ was included with $ax^3+bx^2=cx$, $ax^4+bx^3=cx^2$, $ax^9+bx^8=cx^7$, in a single rule.

We are unable to identify any possible sources of this anonymous author. We only remark that a similar treatment of algebra can be found in part three of Nicholas Chuquet's *Triparty en la science des number,* written in Lyon in 1484 (Chuquet, 1484).

However, the Modena algebra seems to have had almost no influence. We know indeed of only one Italian algebra that can be connected to the same tradition. This is the algebra chapter in the *Libro di ragioni* by Dionigi Gori, written in Siena in 1544 and contained in the manuscript L.IV.22, Biblioteca Comunale of Siena (Gori, 1984).

The first abbacus treatises were printed in Italy at the end of the 15th century, the most important of which was the *Summa de Arithmetica Geometria Proportioni et Proportionalita,* written by Luca Pacioli and printed in Venice in 1494. The *Summa* contains a chapter on algebra that is the first algebra to be printed in Europe.

Pacioli's treatment begins with a translation into the vernacular of the first part of the third section of Chapter 15 from *Liber abaci,* which concerns the six classical equations of first and second degree. Pacioli then makes some interesting remarks on the way the unknown is to be chosen in the solution of problems. After remarking that from the six classical cases it is possible to draw an infinite number of other cases of higher degree, Pacioli says that until then there were no general rules for equations like $ax^4+bx^2=cx$, $ax^4+cx=bx^2$, but concludes that perhaps the rules might be found in the future. Pacioli's treatment of algebra does not represent the deep, rich and complex tradition of investigation into the solutions of trans-quadratic equations that we have examined. We believe that the following developments in algebra in Italy were influenced more by the rich handwritten tradition we have described than by Pacioli's treatise.

6. Conclusions

From the analysis we have made of algebra treatises contained in the Italian manuscripts of the fourteenth century that have come down to us, it emerges that in the first part of the century there were various traditions different from al-Khwarizmi and Fibonacci. In particular, we have picked out three different traditions that we name by their first exponents: Gerardi, Jacopo and Dardi. The texts written by the first two authors are characterized by the presence of equations of degree greater than two, and by the use of algebra also to solve mercantile problems, while Dardi's treatise containing a list of 198 equations is more theoretical. Actually, we know of no previous Arabic or Latin text with these characteristics, so the problem of the origins of the Italian algebraic tradition remains open. In the last part of the 14^{th} century, we find instead a merging of all the above-quoted trends, and also a revival of interest in al-Khwarizmi and Fibonacci. In the following century, the patterns of the algebra chapters in the Italian abbacus treatises show no novelties except for Modena algebra, but the 15^{th} century abbacists considered the knowledge of algebra very important for determining their ability. The importance given to this discipline is also testified by the long historical chapters contained in the manuscripts L.IV.21, Pal. 573 and Ott. lat. 3307. The great attention paid to algebra by Italian abbacists of the fourteenth and fifteenth centuries, and their investigations into the solution of third and fourth degree equations, surely contributed to the diffusion of algebra and paved the way for the great achievements of the next century, when Scipione dal Ferro, Niccolò Tartaglia, Girolamo Cardano and Ludovico Ferrari found the rules to solve these equations.

APPENDIX[20]

Vat. Lat. 4826 (c. 1450), Biblioteca Apostolica Vaticana
Jacopo de Florentia, *Tractatus algorismi* (1307) ff. 1-59
<Algebra> ff. 36v-45v.

Magl. Cl. XI, 87 (1328, d), Biblioteca Nazionale, Firenze
Paolo Gerardi, *Libro di ragioni*, ff. 1-70
Le regole dela cosa, ff. 63r-70r.

Ms. 1754 (c. 1330), Biblioteca Statale, Lucca
Libro di molte ragioni d'abaco, ff. 1-87
Regola della chosa, ff. 50r-52r
<Algebra> ff. 81r-81v.

Ricc. 2252 (s. XIV), Biblioteca Riccardiana, Firenze
(i) *Tractato dell'aritmetica* (c.1320), ff. 1-71
Regole della cosa, ff. 25r-30r
(ii) *Regole dell'algebra* (c.1400), ff. 160r-169r.

Ricc. 2263 (s. XIV), Biblioteca Riccardiana, Firenze
Trattato dell'alcibra amuchabile (c.1365), ff. 24r-50v.

Plut. 30, 26 (1370, d), Biblioteca Mediceo-Laurenziana, Firenze
Giovanni de' Danti, *Tractato dell'algorismo*, ff. 1r-28r.
Per regola del'argibra e per modo di propositione, ff. 20v-21v.

L. IX. 28, (1384, d), Biblioteca Comunale, Siena
Gilio, *<Aritmetica e geometria>* ff. 1-216
Regole della cosa, ff. 37r-45r.

Fond. prin. II. III. 198 (s. XIV), Biblioteca Nazionale, Firenze
(i) *Libro d'insegnare arismetrica* (1390, d), ff. 3r-59v
Delle regole delle cose, ff. 47r-55r
(ii) *Liber de algiebra e almuchabila* (c. 1390), ff. 86r-107v.

Fond. fond. II. V. 152 (c. 1390), Biblioteca Nazionale, Firenze
Tratato sopra l'arte dell'arismetrica, ff. 1r-180
<Calcolo algebrico>, ff. 145r-153r
Le reghole della cosa, ff. 153v-166r
<problemi risolti con l'algebra>, ff. 169v-180v.

Conv. Sopp. G. 7. 1137 (c. 1395), Biblioteca Nazionale, Firenze
Libro delle ragioni d'abaco, ff. 1r-253r
<Algebra>, ff. 107r-109v
Reghole dell'alcibra mochabile, ff. 144r-151v.

Chigi M. VIII. 170 (c. 1395) Biblioteca Apostolica Vaticana
<Dardi>, *Aliabraa-Argibra*, ff. 1r-112r.

2Qq E 13 (1398, d), Biblioteca Comunale, Palermo
Libro merchatantesche, ff. 1r-76v
L'algibra mochabile, ff. 39v-50v.

MA 334 (c.1400) Biblioteca Civica 'Angelo Mai', Bergamo
III. <Maestro Nicheletto. *Zibaldone di matematica*>
(i) (63r-66r) Le chonpoxiçion del açebra
(ii) (78r-77v) Li chapitoli del Açebra.

Magl. Cl. XI, 120 (c.1400), Biblioteca Nazionale, Firenze
(i) <Capitoli d'algebra>, ff. 1r-7r
(ii) M°Antonio da Firenze, Regole del'arzibra, ff. 7v-10v
(iii) <Algebra>, ff. 13r-30v.

Pal . 312 (c.1400), Biblioteca Palatina, Parma
Anonimo, *Libro di conti e mercatantie*, ff. 1-60
Le regole e asempri della Gibra mocabile, ff. 43v-52r

Urb. Lat. 291 (c.1400) Biblioteca Vaticana**,**
Anonimo,< *Trattato di algebra e geometria*>
Fiore de fiori*,* 1r-33v
Algebra e almucabala, 34r-

Pal. 573 (c.1460), Biblioteca Nazionale, Firenze
Anonimo, *Trattato di praticha d'arismetricha*, ff. 1-492
La diecima parte di questo trattato dicente della regola d'algebra amuchabale,
ff. 391r-489v.

L. IV. 21 (1463, d), Biblioteca Comunale, Siena
M°Benedetto, *Trattato di praticha d'arismetricha*, ff. 1-506
El tredecimo libro di questo trattato nel quale si contiene chome e in che modo s'asolvono e chasi per la regola de algebra amuchabale, ff.368r-388r

20. A full palaeographic description of the listed manuscripts can be found in Van Egmond, 1980.

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Lo quarto et decimo libro di questo trattato nel quale si dimostrano chasi exenplari alla regola dell'algibra secondo che scrive Maestro Biaggio, ff. 388v-408r El quindecimo libro di questo trattato nel quale si chontenghono chasi d'alquanti maestri antichi, ff. 408v-474v.

Ott. Lat. 3307 (c.1465), Biblioteca Apostolica Vaticana
Trattato di praticha d'arismetricha, ff. 1r-343r
Decima parte contiene alcuna chosa d'algebra, ff. 304r-342v.

Ital. 578 (c.1485), Biblioteca Estense, Modena
Qui comincia l'agibra, ff. 5r-18v

Ricc. 2265 (c. 1490), Biblioteca Riccardiana, Firenze
Raffaello Canacci, *Vilume dell'algibra*, ff. 1r-211r.

Pal. 567 (c. 1495), Biblioteca Nazionale, Firenze
Raffaello Canacci, *La regola dell'algibra,* ff. 1r-90v.

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