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Gossiping in circulant graphs

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Resum *(CAT)*
Investiguem el problema de fer safareig, en el qual els nodes d'una xarxa d'intercomunicació comparteixen informació mitjançant un protocol de comunicació per rondes. Considerem dos tipus de protocols: per rutes disjuntes en vèrtexs, i per rutes disjuntes en arestes. Donem una fita inferior general per la complexitat dels algoritmes de xafarderies en termes de la funció isoperimètrica de la xarxa. Ens centrem en els grafs de Cayley i donem algorismes òptims per subclasses de grafs de Cayley i, en particular, pels grafs circulants.

Abstract *(ENG)*
We investigate the gossiping problem, in which nodes of an intercommunication network share information initially given to each one of them according to a communication protocol by rounds. We consider two types of communication protocols: vertex-disjoint path mode, and edge-disjoint path mode. We give a general lower bound on the complexity of gossiping algorithms in terms of the isoperimetric function of the graph. We focus on Cayley graphs and give optimal algorithms for subclasses of Cayley graphs and, in particular, for circulant graphs.

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1. Introduction and definitions

1.1 Motivation

In this paper, we study the gossiping problem, in which we disseminate information among an intercommunication network. Initially, each node of the network has some private piece of information. The nodes exchange information through the network, in consecutive rounds, where in each round they can receive or send information, with some constraints according to the communication protocol. The information exchange is complete when each node has learned every piece of information. A gossiping algorithm decides at each round who communicates with whom. We want to find an algorithm which completes the exchange of information in a minimal number of rounds.

We study two communication protocols, the vertex-disjoint path (VDP) mode, and the edge-disjoint path (EDP) mode. In these modes, a node can communicate with another node if they are connected by a path. In every round, the gossiping algorithm selects paths between pairs of nodes that communicate with each other. In the VDP mode, the selected paths need to be vertex-disjoint, that is, they do not have any vertex in common. Similarly, in the EDP mode, the selected paths need to be edge-disjoint, that is, they do not have any edge in common. Moreover, a node can communicate with only one other node during one round. We measure the complexity of a gossiping algorithm by the number of rounds it needs to run. We call the gossip complexity of a network the minimal number of rounds needed by any gossiping algorithm to complete the exchange of information. We choose to study VDP and EDP modes because they are at the same time realistic, and powerful enough to achieve relatively fast gossiping. They have been introduced in [1]. They are widely used in real life applications, and have been extensively studied, see [2, 7, 8, 9, 13, 14]. Each of these modes of communication admits two different versions; they can either be a full duplex or a half duplex mode. In the full duplex version, when two nodes communicate with each other along a path, they both send and receive their information at the same time, whereas for the half duplex version, only one node sends its information and the other receives it. Full duplex modes are well suited for undirected graphs, which are the graphs we study in this paper. Therefore, we only deal with the full duplex version of the VDP and EDP modes.

In order to have good gossip complexity, the intercommunication network needs to have good structural properties. That is why we focus on Cayley graphs, and on circulant graphs, which are a subclass of Cayley graphs. These are popular network topologies.

1.2 Synopsis

In Section 2 we give a general lower bound for the gossip complexity of any graph, in terms of the isoperimetric function of the graph. It is a generalization of the lower bound obtained by Klasing [10]. Thanks to this lower bound, we prove that our gossiping algorithms are optimal, up to a $\log\log(n)$ factor, where n is the order of the graph.

In Section 3, we recall some notions and known results on gossiping. In particular we recall the gossip complexity of the hypercube graph, which is one of the best graphs for the gossiping problem, that is, its gossip complexity is less than the gossip complexity of any other graph. Knödel describes an optimal gossiping algorithm for the hypercube in [11]. Therefore, naturally, for graphs that have a similar structure to the hypercube, we try to use a similar gossiping algorithm. Indeed, many graphs embed into the hypercube graphs. We define the concept of embedding in the same Section 3. With this tool, we can simulate the gossiping algorithm of the hypercube graph in many other graphs, and get almost optimal algorithms. This is done for cube-connected cycles and butterfly networks by Hromkovic, Klasing and Stöhr [7], for the grid by Hromkovic, Klasing, Stöhr and Wagener [8], or more recently for circulant graphs (whose definition is given in Section 4) by Mans and Shparlinski [15]. These are standard graphs, structurally close to the hypercube. In [15], the gossiping algorithm given by the authors only works for the subclass of circulant graphs whose generator set is of size two.

In Section 4, we give an algorithm which works in almost optimal time for a wider subclass of circulant graphs, in particular for circulant graphs whose generator set is of any size. Finally, we extend the gossiping algorithm working for the circulant graphs to a more general class of Cayley graphs.

1.3 Definitions and notations

We recall here some basic definitions of graph theory. Throughout, $[n]$ denotes the set $\{1, \dots, n\}$. Let G be a *graph*. Except if mentioned explicitly, the graphs we consider are undirected and connected. We denote by $V(G)$ its set of vertices, and by $E(G)$ its set of edges. A *path* in G is a sequence of distinct vertices $u_0 u_1 \cdots u_l$, where for all i in $\{0, \dots, l-1\}$, $\{u_i, u_{i+1}\}$ is an edge of G. The length of the path, l, may be equal to 0, in which case the path is reduced to the vertex u_0. We denote by $P(G)$ the set of all paths in G. We say that an edge $e \in E(G)$ *belongs* to the path $p \in P(G)$ if we have $p = u_0 u_1 \cdots u_l$ and $e = \{u_i, u_{i+1}\}$, for some $i \in \{0, \dots, l-1\}$. Similarly, u *belongs* to p if we have $u = u_i$ for some $i \in \{0, \dots, l\}$. We say that two paths p and p' of $P(G)$ are *vertex-disjoint* if and only if there is no vertex u of $V(G)$ that belongs to p *and* p'. Similarly, they are *edge-disjoint* if and only if there is no edge e in $E(V)$ belonging to p *and* p'. For any path $p = u_0 \cdots u_l \in P(G)$, we call the vertices u_0 and u_l the extremities of p.

Here we give the definition of a broadcast algorithm, an accumulation algorithm, and a gossiping algorithm, which are of fundamental importance. A *communication algorithm A for the VDP mode (resp. EDP mode) in the graph G* is defined by a sequence of $t(A)$ rounds $E_1, E_2, \dots, E_{t(A)}$, with a round E_i being a set of pairwise vertex-disjoint paths of G (resp. pairwise edge-disjoint paths of G). The integer $t(A)$ is the *complexity* of A. Moreover, each node of $V(G)$ can not be the extremity of more than one path of E_i. For every vertex $v \in V(G)$ and for all $r \in \{0, \dots, t(A)\}$, we denote by $I_v(r)$ the set of information known to v after the r-th round of algorithm A. $I_v(r)$ is defined recursively by $I_v(0) = \{v\}$, and $I_v(r) = I_v(r-1) \cup I_w(r-1)$ if there exists a path p of the form $p = v, \dots, w$ or $p = w, \dots, v$ in E_r; $I_v(r) = I_v(r-1)$ otherwise.

A is a *broadcast algorithm* for the set of vertices $U \subseteq V(G)$ if for all v in $V(G)$, we have $U \subseteq I_v(t(A))$. Similarly, A is an *accumulation algorithm* for the set of vertices $U \subseteq V(G)$ if $\bigcup_{u \in U} I_u(t(A)) = V(G)$. A is a *gossiping algorithm* if for all v in $V(G)$, we have $I_v(t(A)) = V(G)$. In other words, a gossiping algorithm performs communication between the nodes in such a way that at the end of the algorithm, every node knows the secret of every other node. We call the *gossip complexity* of a graph the minimal number of rounds to achieve the gossiping in this graph. More precisely, if we denote by $\mathrm{A}^{\mathrm{G}}_{\mathrm{VDP}}$ the set of all gossiping algorithms for the VDP mode in G (resp. $\mathrm{A}^{\mathrm{G}}_{\mathrm{EDP}}$ for the EDP mode), the gossip complexity of G for the VDP mode, $\mathrm{g}_{\mathrm{VDP}}(G)$, is

$$\mathrm{g}_{\mathrm{VDP}}(G) = \min_{A \in \mathrm{A}^{\mathrm{G}}_{\mathrm{VDP}}} \{t(A)\}, \qquad (\text{resp. } \mathrm{g}_{\mathrm{EDP}}(G) = \min_{A \in \mathrm{A}^{\mathrm{G}}_{\mathrm{EDP}}} \{t(A)\}).$$

We define similarly the broadcast complexity and the accumulation complexity for a set of vertices $U \subseteq V(G)$ of a graph G, which we denote, respectively, by $\mathrm{b}_{\mathrm{VDP}}(G, U)$ and $\mathrm{a}_{\mathrm{VDP}}(G, U)$ for the VDP mode, and $\mathrm{b}_{\mathrm{EDP}}(G, U)$ and $\mathrm{a}_{\mathrm{EDP}}(G, U)$ for the EDP mode.

2. A lower bound for the gossip complexity

In this section, we give a general lower bound on the gossip complexity of any graph for the VDP and EDP modes. In [8], J. Hromkovic et al. prove a lower bound on the gossip complexity of any graph, depending on its bisection width. We can actually prove a more general lower bound, which depends on the isoperimetric number of the graph. We first give the definition of the two notions of bisection width and isoperimetric number of a graph. Then, in Theorem 2.1, we generalize the lower bound of J. Hromkovic et al. [8]. In [10], R. Klasing gives a slightly better lower bound on the gossip complexity of a graph in terms of its bisection width. It is also possible to generalize this lower bound, and obtain Theorem 2.2.

Let $G = (V, E)$ be a graph. For every $U \subseteq V$, we denote by $\partial_{\text{in}}(U)$ the *inner vertex-boundary of* U, defined by

$$\partial_{\text{in}}(U) = \big\{u \in U : \exists v \in V \setminus U, \{u, v\} \in E\big\}.$$

Similarly, we denote by $e(U)$ the *inner edge-boundary of* U, defined by

$$e(U) = \big\{\{u, v\} \in E : u \in U, v \in V \setminus U\big\}.$$

The *vertex bisection width* $\text{vbw}(G)$ *of* G is defined by

$$\text{vbw}(G) = \min\left\{|\partial_{\text{in}}(U)| : U \subset V, \left\lfloor\frac{|V|}{2}\right\rfloor \le |U| \le \left\lceil\frac{|V|}{2}\right\rceil\right\}.$$

Similarly, the *edge bisection width* $\text{ebw}(G)$ of G is defined by replacing $|\partial_{\text{in}}(U)|$ by $|e(U)|$ in the above definition. More generally, the *vertex isoperimetric number* of G is

$$\text{vi}(G, t) = \min\big\{|\partial_{\text{in}}(U)| : U \subset V, |U| = t\big\},$$

and the *edge isoperimetric number* $\text{ei}(G, t)$ is obtained by replacing $|\partial_{\text{in}}(U)|$ by $|e(U)|$ in the above definition. Intuitively, the isoperimetric number tells us if there is a bottleneck in a given graph, which would imply a high gossip complexity.

We can now state the theorem for the VDP mode.

Theorem 2.1. *Let* $G = (V, E)$ *be a graph and* (V_1, V_2) *a partition of its vertex set into parts of size* $|V_1| = n_1$ *and* $|V_2| = n_2$. *Let* $k = |\partial_{\text{in}}(V_1)|$ *and* $l = |e(V_1)|$. *Then*

$$g_{\text{VDP}}(G) \;\ge\; \log(n_1 n_2) - \log(k) - \log(\log(n_1)) - 2$$

and

$$g_{\text{EDP}}(G) \;\ge\; \log(n_1 n_2) - \log(l) - \log(\log(n_1)) - 2.$$

In particular, the inequality holds for $k = \max_t \text{vi}(G, t)$, *and* $l = \max_t \text{ei}(G, t)$.

We give the proof of Theorem 2.1 for the VDP mode. The proof for the EDP mode is obtained similarly, replacing the inner boundary $\partial_{\text{in}}(V_1)$ by the edge boundary $e(V_1)$.

Proof. Let $G = (V, E)$ be a graph, (V_1, V_2) a partition of its vertex set into parts of size $|V_1| = n_1$ and $|V_2| = n_2$, and $k = |\partial_{\text{in}}(V_1)|$. The idea of the proof is to estimate how much information can flow from V_1 to V_2.

Let $A = E_1, \dots, E_{t(A)}$ be a gossiping algorithm for G. For all $r \in \{0, \dots, t(A)\}$, $I_v(r)$ is the information known by v after the rth round, as defined in Section 1.3. We define $I_v^1(r)$ as $I_v^1(r) := I_v(r) \cap V_1$, and $I(r) := \bigcup_{v \in V_2} I_v^1(r)$. The value $I(r)$ represents the information that has gone from V_1 to V_2 during the first r rounds. Since A is a gossiping algorithm, every node $v \in V_2$ knows the information of all nodes in V_1 after $t(A)$ rounds, that is, it must be

$$I(t(A)) \;\geq\; |V_1| \cdot |V_2| = n_1 n_2. \tag{1}$$

Now we give an upper bound on $I(t(A))$. For all $r \in \{0, \dots, t(A)\}$, we define

$$\widehat{I}(r) := \bigcup_{v \in \partial_{\text{in}}(V_1)} I_v^1(r).$$

The value $\widehat{I}(r)$ represents the amount of information that can go from V_1 to V_2 in round r. We observe that the amount of information of a node can be at most doubled in each round. That is, for all v in V_1 and for all r in $\{0, \dots, \lfloor \log(n_1) \rfloor\}$, we have $I_v^1(r) \leq 2^r$. Therefore, we have

$$\widehat{I}(r) \;\leq\; k \min(2^r, n_1). \tag{2}$$

The amount of information from V_1 already present in V_2 in round r can be at most doubled in round $r+1$:

$$I(r+1) \;\leq\; 2I(r) + \widehat{I}(r). \tag{3}$$

Combining equations (2) and (3), we get:

- For all $0 \leq r \leq \log(n_1)$, $\quad I(r+1) \leq 2I(r) + k2^r$.
- For all $\log(n_1) \leq r$, $\quad I(r+1) \leq 2I(r) + kn_1$.

By induction, $I(r) \leq r \cdot k \cdot 2^{r-1}$ for all $0 \leq r \leq \log(n_1)$. Moreover, for all $r > \log(n_1)$, we obtain that $I(r) \leq k \cdot 2^{r-1}(\log(n_1) + 1) - \frac{n_1 k}{2}$. In particular, for $r = t(A)$, equation (1) yields

$$n_1 n_2 \;\leq\; I(r) \;\leq\; k \cdot 2^{t(A)-1}(\log(n_1) + 1).$$

Therefore, by taking logarithms to both sides of the inequality, we get

$$\log(n_1 n_2) - \log(k) - \log(\log(n_1)) - 2 \leq t.$$

The result for the EDP mode can be obtained similarly. □

In [10], R. Klasing proves that we can improve the lower bound of J. Hromkovic et al. [8]. More precisely, he shows that we have a gossip complexity of at least $2\log(n) - \log(k) - \log(\log(k)) + O(1)$ for any graph of order n and bisection width k. We can generalize the lower bound in [8] to obtain the following theorem.

Theorem 2.2. *Let $G = (V, E)$ be a graph and (V_1, V_2) a partition of its vertex set, of size $|V_1| = n_1$ and $|V_2| = n_2$. We denote by $k = |\partial_{\text{in}}(V_1)|$ and $l = |e(V_1)|$. Then*

$$g_{\text{VDP}}(G) \geq \log(n_1 n_2) - \log(k) - \log(\log(k)) + O(1)$$

and

$$g_{\text{EDP}}(G) \geq \log(n_1 n_2) - \log(l) - \log(\log(l)) + O(1).$$

In particular, the inequality holds for $k = \max_t \text{vi}(G, t)$, and $l = \max_t \text{ei}(G, t)$. □

We omit the proof due to lack of space. It can be found in [6].

3. Basic examples, embeddings and three phase algorithms

In this section we recall the gossip complexity of some basic graphs, including the hypercube. Then we present the concept of embedding, thanks to which we can extend the gossiping algorithm for the hypercube to other similar graphs. Finally we present the so-called three phase algorithm strategy which will prove useful later on.

3.1 Basic examples

We first give a general lower bound for the gossip complexity in any graph.

Lemma 3.1. *For any graph $G = (V, E)$, for all v in V and for all $0 \leq r \leq \lfloor \log(n) \rfloor$, $|I_v(r)| \leq 2^r$. In particular,*

$$g_{\mathrm{VDP}}(G) \ \geq \ g_{\mathrm{EDP}}(G) \ \geq \ \log n.$$

Proof. By induction on r. Let $v \in V$. For $r = 0$ we have $|I_v(0)| = 1$. For all $r < \lfloor \log(n) \rfloor$, either there exists w in V such that $|I_v(r+1)| = |I_v(r) \cup I_w(r)| \leq 2^{r+1}$, or $|I_v(r+1)| = |I_v(r)| \leq 2^r \leq 2^{r+1}$. $\square$

The hypercube is a widely used graph which is known to have good communication properties, especially for the gossiping problem. We recall its definition here.

Definition 3.2. For all $k \geq 2$, $d \geq 1$, the k-ary *hypercube* of dimension d, $H(k, d)$, is the graph defined by the set of vertices $V = \{0, \dots, k-1\}^d$, and the set of edges E such that $\forall \alpha = a_1 \cdots a_n$, $\beta = b_1 \cdots b_n \in V$, $\{\alpha, \beta\} \in E$ if and only if $\exists i \in \{1, \dots, d\}$ such that $b_i \neq a_i$ and, $\forall j \in \{1, \dots, d\} \backslash \{i\}$, $b_j = a_j$.

Theorem 3.3 (Hromkovic, Klasing, Stöhr, [7])**.** *For all $k \geq 2$ and $d \geq 1$,*

$$d\lceil \log(k) \rceil \ \leq \ g_{\mathrm{EDP}}(H(k,d)) \ \leq \ g_{\mathrm{VDP}}(H(k,d)) \ \leq \ d(\lceil \log(k) \rceil + 1).$$

According to this theorem, the hypercube is the best graph for gossiping, together with the complete graph. For the latter, we have the following result by Knödel [11].

Theorem 3.4 (Knödel [11])**.** *For all $n \in \mathbb{N}$, let K_n be the complete graph of size n. Then*

$$\lceil \log(n) \rceil \ \leq \ g_{\mathrm{VDP}}(K_n) \ = \ g_{\mathrm{EDP}}(K_n) \ \leq \ \lceil \log(n) \rceil + 1.$$

3.2 Embeddings

We have seen that we can gossip in a really efficient way in the hypercube. In many other graphs, we can use similar algorithms to gossip efficiently. More generally, many graphs "contain" other subgraphs in which we know how to gossip efficiently. In order to transfer results from the subgraph to the super-graph, we use the concept of embedding.

We give the definitions of an embedding, its load, and its vertex and edge-congestion, which can be found in Kolman [12]. We also introduce new definitions, such as vertex and edge-congestion *for an algorithm A*, which will be useful in the next section.

- Let G and H be two undirected graphs. An *embedding* of the graph G into the graph H is a mapping f of the vertices of G into the vertices of H, together with a mapping g of edges of G into paths in H, such that g assigns to each edge $\{u, v\} \in E(G)$ a path from $f(u)$ to $f(v)$ in $P(H)$.

- The *load* of the embedding is the maximum number of vertices of G mapped to a single vertex of H:

$$\mathrm{load}(f, g) \;=\; \max_{v \in V(H)} \left|\{u \in V(G) : f(u) = v\}\right|.$$

  The *edge-congestion* $\mathrm{e_{cong}}(f, g)$ is defined by

$$\mathrm{e_{cong}}(f, g) \;=\; \max_{e \in E(H)} \left|\{e' \in E(G) : e \text{ belongs to } g(e')\}\right|.$$

  Similarly, the *vertex-congestion* $\mathrm{v_{cong}}(f, g)$ is defined by

$$\mathrm{v_{cong}}(f, g) \;=\; \max_{u \in V(H)} \left|\{e' \in E(G) : u \text{ belongs to } g(e')\}\right|.$$

- We do not need all paths in $g(E(G))$ to be pairwise vertex or edge disjoint, because not all edges are used at the same time by a communication algorithm. That is why we introduce a weaker notion. Let $A = E_1, E_2, \ldots, E_{t(A)}$ be a communication algorithm.

  For all $e \in E(G)$ and for all $r \in \{1, \ldots, t(A)\}$, we say that e is *active* in round r if and only if there exists a path $p \in E_r$ such that e belongs to p. We denote by $\mathrm{AE}(r)$ the set of *active edges* in round r.

  In the same way, for all $u \in V(G)$, we say that u is active in round r if and only if there exists a path $p \in E_r$ such that u belongs to p. We denote by $\mathrm{AV}(r)$ the set of *active vertices* in round r.

  We define the *vertex congestion for algorithm A*, ${\mathrm{v_{cong}}}^A(f, g)$, by

$${\mathrm{v_{cong}}}^A(f, g) \;=\; \max_{r \in \{1,\ldots,t(A)\}, u \in V(H)} \left|\{e' \in \mathrm{AE}(r) : u \text{ belongs to } g(e')\}\right|.$$

  Similarly,

$${\mathrm{e_{cong}}}^A(f, g) \;=\; \max_{r \in \{1,\ldots,t(A)\}, e \in E(H)} \left|\{e' \in \mathrm{AE}(r) : e \text{ belongs to } g(e')\}\right|.$$

  Finally, we define $\mathrm{load}^A(f, g)$ by

$$\mathrm{load}^A(f, g) \;=\; \max_{r \in \{1,\ldots,t(A)\}, v \in V(H)} \left|\{u \in \mathrm{AV}(r) : f(u) = v\}\right|.$$

With the above definitions we can now state our theorem. This theorem in implicit in [7, 8, 9, 15].

Theorem 3.5. *Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two graphs. If A is a gossiping algorithm for G in the VDP mode (resp., EDP mode), which runs in $t(A)$ rounds; and if f,g is an embedding of G into H such that $\mathrm{load}^A(f, g) = 1$, and ${\mathrm{v_{cong}}}^A(f, g) = 1$ (resp. ${\mathrm{e_{cong}}}^A(f, g) = 1$), then we can gossip among the set of vertices $f(V(G))$ in H in less than $t(A)$ rounds in the VDP mode (resp., the EDP mode).*

Proof. We extend the function $g : E(G) \to P(H)$ on the paths of G via

$$g(u_0 u_1 \cdots u_l) \;=\; g(\{u_0, u_1\}) g(\{u_1, u_2\}) \cdots g(\{u_{l-1} u_l\})$$

for all $u_0 u_1 \cdots u_l \in P(G)$, i.e., we concatenate the images of all edges of the path $u_0 u_1 \cdots u_l$.

Let $A = E_1 E_2 \cdots E_{t(A)}$ be the gossiping algorithm of G for the VDP mode. We construct an algorithm A' which performs the gossiping among $f(V(G))$ in H in $t(A)$ rounds as follows: whenever the vertex $u \in V(G)$ communicates with $v \in V(G)$ through the path $p \in P(G)$, $f(u) \in V(H)$ communicates with $f(v) \in V(H)$ through the path $g(p) \in P(H)$. It is well defined because $f(u) \neq f(v)$, since $\mathrm{load}^A(f,g) = 1$. Let $r \in \{1, \dots, t(A)\}$ such that $E_r = \{p_1, \dots, p_l\}$, $l \geq 1$. Since $\mathrm{v_{cong}}^A(f,g) = 1$, $\{g(p_1), \dots, g(p_l)\}$ is a set of vertex-disjoint paths of $P(H)$. At the end of algorithm A', for each vertex $u \in V(G)$,

$$I_{f(u)}(t(A)) \;=\; \bigcup_{v \in V(G), f(v) = f(u)} f(I_v(t(A))),$$

so $I_{f(u)}(t(A)) = f(V(G))$. Therefore A' performs the gossiping among $f(V(G))$ properly for the VDP mode. An analogous argument works for the EDP mode. □

3.3 Three phases algorithm

In most of the hypercube-like graphs, we use a gossiping algorithm that first accumulates the information of the entire graph into a subgraph, then gossip in the subgraph as in the hypercube, and finally broadcast the information to the whole graph. This is called a three-phase algorithm.

Definition 3.6. We say a gossiping algorithm is a *three-phase algorithm* if it performs an accumulation phase, then a gossiping phase, and finally a broadcast phase:

1. **Accumulation phase:** G is divided into connected components (called *accumulation components*), each component containing exactly one accumulation node. This node accumulates the information from the nodes lying in its component.

2. **Gossip phase:** Let $a(G)$ be the set of all accumulation nodes in G. A gossiping algorithm is performed among the nodes in $a(G)$. All nodes in $V(G) - a(G)$ are considered to have no information, and they are only used to build disjoint paths between receivers and senders from $a(G)$.

3. **Broadcast phase:** Every node in $a(G)$ broadcasts the information to its component.

Here we present a useful lemma on the number of rounds needed to accumulate all the information of the path of length $n \in \mathbb{N}^*$ into one vertex at the end of the path.

Lemma 3.7 (Feldmann, Hromkovic, Monien, Madhavapeddy and Mysliwietz, [4])**.** *For all $n \in \mathbb{N}^*$, let P_n be the path of length n, i.e the graph with vertex set $\{0, \dots, n-1\}$ and edge set $E = \{\{i, i+1\}, i \in \{0, \dots, n-2\}\}$. Then*

$$\mathrm{b_{VDP}}(P_n, \{0\}) \;=\; \mathrm{a_{VDP}}(P_n, \{0\}) \;=\; \mathrm{b_{EDP}}(P_n, \{0\}) \;=\; \mathrm{a_{EDP}}(P_n, \{0\}) \;\leq\; \lceil \log(n) \rceil.$$

4. Gossiping in circulant graphs

In this section, we present the main results of this paper. Mans and Shparlinski [15] gave an optimal gossiping algorithm for some circulant graphs whose generator set is of size two. We exhibit a gossiping algorithm for more general circulant graphs whose generator set can be of any size.

We recall here the definitions of Cayley graphs and circulant graphs, which are a particular type of Cayley graphs.

Let $(G, +)$ be an additively written group, and let $S \subseteq G$ be a subset of G. The *Cayley graph* $\Gamma(G, S)$ is the graph with vertex set $V = G$ and set of arcs E such that for all $u, v \in V$, $(u, v) \in E$ if and only if there exists $s \in S$ such that $v = u + s$. If $\Gamma(G, S)$ is to be connected, S must be a generating set of G. If we want $\Gamma(G, S)$ to be undirected, S must be symmetric, i.e. of the form $S = \{\pm s_1, \dots, \pm s_r\}$.

For example, for $k \geq 2$ and $d \geq 1$, the k-ary hypercube of dimension d, $H(k, d)$, is the Cayley graph $\Gamma(\mathbb{Z}_k^d, S)$, with $S = \bigcup_{i\in[d]}\{\pm\lambda \cdot e_i : \lambda \in [k-1]\}$, where e_i is the vector whose coordinates are all zero except for the i-th coordinate, which is 1. We can generalize Theorem 3.3 for the k-ary hypercube of dimension d in the following way.

Theorem 4.1. *Let $d \geq 1$, $k_1, k_2, \dots, k_d \geq 2$. Let $S = \bigcup_{i\in[d]}\{\pm\lambda \cdot e_i : \lambda \in [k_i - 1]\}$. Then*

$$\begin{aligned}\sum_{i\in[d]} \lceil \log(k_i) \rceil &\leq g_{\mathrm{EDP}}\left(\Gamma(\mathbb{Z}_{k_1} \times \mathbb{Z}_{k_2} \cdots \times \mathbb{Z}_{k_d}, S)\right)\\ &\leq g_{\mathrm{VDP}}\left(\Gamma(\mathbb{Z}_{k_1} \times \mathbb{Z}_{k_2} \cdots \times \mathbb{Z}_{k_d}, S)\right)\\ &\leq \sum_{i\in[d]} \left(\lceil \log(k_i) \rceil + 1\right).\end{aligned}$$

Proof. The lower bound comes from Lemma 3.1. For the upper bound, we use Algorithm 1.

Algorithm 1 Gossip$\left(\Gamma\left(\prod_{i=1}^{d} \mathbb{Z}_{k_i}, S\right)\right)$

for $i = 1$ to d **do**
  for all $\alpha \in \prod_{j=1}^{i-1} \mathbb{Z}_{k_j}$ and $\beta \in \prod_{l=i+1}^{d} \mathbb{Z}_{k_l}$ **do in parallel**
    gossip in $L_{\alpha,\beta} = \{\alpha m \beta : m \in \{0, \dots, k_i - 1\}\}$
  end do in parallel
end for

procedure Gossip in $L_{\alpha,\beta} = \{\alpha m \beta : m \in \{0, \dots, k_i - 1\}\}$
  do in parallel
    Gossip in $\{\alpha m \beta : m \in \{0, \dots, \left\lfloor \frac{k_i}{2} \right\rfloor - 1\}\}$ **and**
    Gossip in $\{\alpha m \beta : m \in \{\left\lfloor \frac{k_i}{2} \right\rfloor, \dots, k_i - 1\}\}$
  end do in parallel
  for $l = 0$ to $\left\lfloor \frac{k_i}{2} \right\rfloor - 1$ **do in parallel**
    exchange information between $\alpha l \beta$ and $\alpha(m - l - 1)\beta$
  end do in parallel
end procedure

For all $i \in [d]$, and for all $\alpha \in \prod_{j=1}^{i-1} \mathbb{Z}_{k_j}$, $\beta \in \prod_{l=i+1}^{d} \mathbb{Z}_{k_l}$, the subgraph $L_{\alpha,\beta}$ induced by the set of vertices $\{\alpha m \beta : m \in \{0, \dots, k_i - 1\}\}$ is a clique, so we can gossip in $L_{\alpha,\beta}$ with the procedure Gossip of Algorithm 1. This procedure uses the algorithm of the complete graph K_{k_i}, which works in at most $\lceil \log(k_i) \rceil + 1$ rounds according to Theorem 3.4. So the total number of rounds needed to gossip in $\Gamma(\mathbb{Z}_{k_1} \times \mathbb{Z}_{k_2} \cdots \times \mathbb{Z}_{k_d}, S)$ is at most $\sum_{i\in[d]}(\lceil \log(k_i) \rceil + 1)$. □

When $G = \mathbb{Z}_n$ is the cyclic group and $S = -S \subseteq \mathbb{Z}_n$ is a symmetric subset of $\mathbb{Z}_n$, the Cayley graph $\Gamma(\mathbb{Z}_n, S)$ is called a *circulant graph*, and will be denoted by $C(n, S)$. In the next theorem, we give a lower bound of the gossip complexity for any circulant graph using Theorem 2.2. The result was obtained by Mans and Shparlinski [15].

Theorem 4.2. *For all $n,r \in \mathbb{N}^*$, for all $S = \{\pm s_1, \dots, \pm s_r\} \subseteq \mathbb{Z}_n$ such that $s_1 \leq s_2 \leq \dots \leq s_r$,*

$$g_{\mathrm{VDP}}(C(n,S)) \ \geq \ 2\log(n) - \log(s_r) - \log(\log(s_r)) + O(1)$$

and

$$g_{\mathrm{EDP}}(C(n,S)) \ \geq \ 2\log(n) - \log(rs_r) - \log(\log(rs_r)) + O(1).$$

Proof. We number the nodes from 0 to $n-1$. Let $V_1 = \{0, \dots, \lfloor \frac{n}{2} \rfloor - 1\}$ and $V_2 = \{\lfloor \frac{n}{2} \rfloor, \dots, n-1\}$. Then $\partial^G_{\mathrm{in}}(V_1) \subseteq \{0, \dots, s_r - 1\} \cup \{\lfloor \frac{n}{2} \rfloor - s_r, \dots, \lfloor \frac{n}{2} \rfloor - 1\}$, because for all $u \in \{s_r, \dots, \lfloor \frac{n}{2} \rfloor - s_r - 1\}$, and all v in V_2, we have that $v - u > s_r \bmod n$, so $\{u, v\} \notin E$. Thus, $|\partial^G_{\mathrm{in}}(V_1)| \leq 2s_r$. Theorem 2.2 then yields

$$\begin{aligned} g_{\mathrm{VDP}}(C(n,S)) \ &\geq \ \log\left(\left\lfloor \frac{n}{2} \right\rfloor \cdot \left\lceil \frac{n}{2} \right\rceil\right) - \log(s_r) - \log(\log(s_r)) + O(1) \\ &\geq \ 2\log(n) - \log(s_r) - \log(\log(s_r)) + O(1). \end{aligned}$$

It is easy to check that $|\mathrm{e}(V_1)| \leq 2\sum_{i=1}^{r} s_i \leq 2rs_r$; for further details, see [15]. Theorem 2.2 yields

$$\begin{aligned} g_{\mathrm{EDP}}(C(n,S)) \ &\geq \ \log\left(\left\lfloor \frac{n}{2} \right\rfloor \cdot \left\lceil \frac{n}{2} \right\rceil\right) - \log(2rs_r) - \log(\log(2rs_r)) + O(1) \\ &\geq \ 2\log(n) - \log(rs_r) - \log(\log(rs_r)) + O(1). \end{aligned}$$

This concludes the proof. □

For particular instances of S, we know an algorithm which matches the previous lower bound. For instance, when $S = \{\pm 1, \pm n^{1/r}, \dots, \pm n^{(r-1)/r}\}$, then $C(n, S)$ admits the grid $Gr(n^{\frac{1}{r}}, r)$ as a spanning subgraph. So we can apply the algorithm of the grid of [8] which matches the lower bound. But in the general case, we do not know whether the previous lower bound is tight. In this paper, we find an algorithm for a general class of circulant graphs, which (almost) matches the lower bound. Such an approach can be found in [15], where B. Mans and I. E. Shparlinski find an (almost) optimal gossiping algorithm for circulant graphs where $r = 2$. More precisely, they prove that if $S = \{\pm 1, \pm s_2\}$ and $s_2 \leq 2\lfloor p/s_2 \rfloor$, then the lower bound of Theorem 4.2 is tight. In fact, they give an algorithm which performs in (almost) optimal time. We have generalized this approach to arbitrary r.

Theorem 4.3. *Let $n \in \mathbb{N}^*$, and $C(n, S)$ be a circulant graph with generating set $S = \{\pm s_1, \dots, \pm s_r\}$. If $s_1 = 1$, $s_1 < s_2 < \dots < s_r$ and $\lceil \frac{s_{i+1}}{s_i} \rceil \leq 2\frac{n}{s_r}$ for all $i \in [r-1]$, then*

$$2\log(n) - \log(s_r) + 2r \ \geq \ g_{\mathrm{VDP}}(C(n,S)) \ \geq \ 2\log(n) - \log(s_r) - \log(\log(s_r)) + O(1)$$

and

$$2\log(n) - \log(s_r) + 2r \ \geq \ g_{\mathrm{EDP}}(C(n,S)) \ \geq \ 2\log(n) - \log(rs_r) - \log(\log(rs_r)) + O(1).$$

Proof. We exhibit a three phase algorithm that works in at most $2\log(n) - \log(s_r) + 2r$ rounds for the VDP mode.

Accumulation phase:

We number the nodes of $C(n, S)$ from 0 to $n-1$, and identify each node with its number. We choose the accumulation nodes $a(G)$ to be $\{0, \dots, s_r - 1\}$, and the accumulation components to be

$$A_j = \{i \in \mathbb{Z}_n : i = j \bmod s_r\}, \qquad \text{for all } j \in \{0, \dots, s_r - 1\}.$$

All accumulation components are of size at most $\lfloor (n-1)/s_r \rfloor + 1$. So the accumulation phase takes at most $\lceil \log \left(\lfloor (n-1)/s_r \rfloor + 1 \right) \rceil$ many rounds, which in turn is at most $\log(n/s_r) + 1$.

Gossip phase:

To simplify the proof, we suppose that

$$\frac{s_{i+1}}{s_i} = q_i \in \mathbb{N} \qquad \text{for all } i \in [r-1]. \tag{4}$$

Let $S' = \bigcup_{i \in [r-1]} \{\pm \lambda \cdot e_i : \lambda \in [q_i - 1]\}$. The Cayley graph $\Gamma\left(\mathbb{Z}_{q_1} \times \mathbb{Z}_{q_2} \times \cdots \times \mathbb{Z}_{q_{r-1}}, S'\right)$ is embedded into $C(n, S)$, where $f : \prod_{i=1}^{r-1} \mathbb{Z}_{q_i} \to \mathbb{Z}_n$ is defined for all $a_1 a_2 \cdots a_{r-1} \in \prod_{i=1}^{r-1} \mathbb{Z}_{q_i}$ by

$$f(a_1 a_2 \cdots a_{r-1}) = \sum_{i=1}^{r-1} a_i \cdot s_i \in \mathbb{Z}_n.$$

Let $u = a_1 \cdots a_{r-1}$ be a node in $\prod_{l=1}^{r-1} \mathbb{Z}_{q_l}$, and $v = a_1 \cdots a_{i-1} a'_i a_{i+1} \cdots a_{r-1}$, with $a_i < a'_i$. Then $g(u, v)$ is defined to be the path which goes from vertex $f(u) = \sum_{j \in [r-1]} a_j \cdot s_j$ to $\sum_{j \in [r-1]} a_j \cdot s_j + t_{a,b} \cdot s_r$ through $t_{a,b}$ chords $+s_r$, then to vertex

$$\sum_{j \in [r-1] \setminus \{i\}} a_j \cdot s_j + a'_i \cdot s_i + t_{a,b} \cdot s_r$$

through $a'_i - a_i$ chords $+s_i$, and finally to vertex

$$\sum_{j \in [r-1] \setminus \{i\}} a_j \cdot s_j + a'_i \cdot s_i$$

through $t_{a,b}$ chords $-s_r$. We choose $t_{a,b} = \left\lfloor \frac{b-a}{2} \right\rfloor$. For an illustration of the embedding (f, g), see Figure 1.

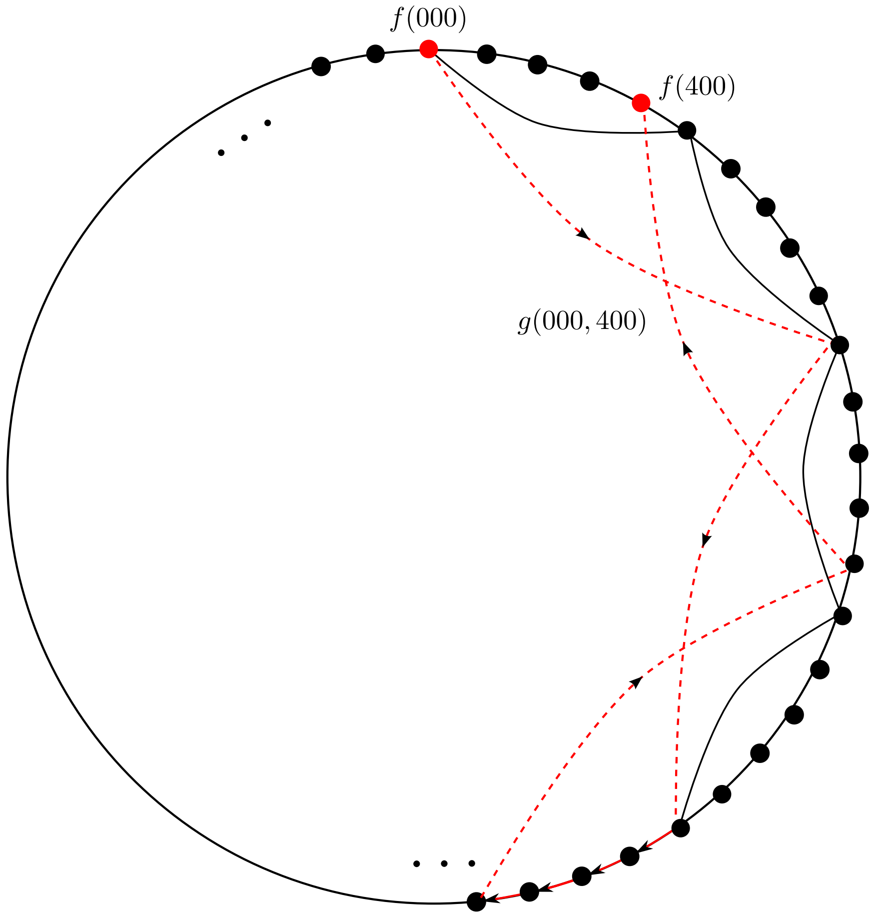
According to Theorem 3.4, there is a gossiping algorithm A for $\Gamma\left(\prod_{i=1}^{r-1} \mathbb{Z}_{q_i}, S'\right)$ for the VDP mode, which works in at most $\sum_{i=1}^{r-1} \left(\lceil \log(q_i) \rceil + 1 \right) \le \log(s_r) + 2(r-1)$ rounds. It is easy to check that the load of the embedding f, g for algorithm A is one. We show that $\mathrm{v_{cong}}^A(f, g) = \mathrm{e_{cong}}^A(f, g) = 1$. In Algorithm A, in each round, the exchanges of information are of the form

$$a = a_1 \cdots a_{r-1} \in \prod_{i=1}^{r-1} \mathbb{Z}_{q_i} \text{ exchanges its information with } a' = a_1 \cdots a_{i-1} a'_i a_{i+1} \cdots a_{r-1}, \text{ with } a_i < a'_i.$$

Suppose that in the same round,

$$b = b_1 \cdots b_{r-1} \text{ exchanges its information with } b' = b_1 \cdots b_{i-1} b'_i b_{i+1} \cdots b_{r-1}, \text{ with } b_i < b'_i.$$

By the construction of algorithm A (see Algorithm 1), $b_i < a_i$ and $a'_i < b'_i$, or $a_i < b_i$ and $b'_i < a'_i$. So $t_{a,a'} \neq t_{b,b'}$, and $\mathrm{v}^{\mathrm{A}}_{\mathrm{cong}}(f, g) = 1$. Therefore, by Theorem 3.5, there is an algorithm A' that performs the gossiping in $f\left(\prod_{i=1}^{r-1} \mathbb{Z}_{q_i}\right) = \{0, \dots, s_r - 1\}$ in at most $\log(s_r) + 2(r-1)$ rounds for the VDP mode. A' is described in Algorithm 2.



$\Gamma(\mathbb{Z}_5 \times \mathbb{Z}_2)$ is embedded into $C(n, S)$.

Figure 1: $C(n, S)$, with $S = \{\pm 1, \pm 5, \pm 10\}$

Broadcast phase:

By symmetry, the broadcast phase takes at most $\log(p/s_r) + 1$ rounds, just like the accumulation phase. Therefore the total number of rounds needed to gossip in $C(n, S)$ is at most $2\log(n) - \log(s_r) + 2r$.

In the case where we do not assume condition (4) anymore, we need to slightly modify Algorithm 2, but the essential arguments remain the same. This concludes the proof of Theorem 4.3. □

We have exhibited an algorithm that matches the lower bound on the gossip complexity for some circulant graphs (up to a $\log(\log(n))$ factor, n being the size of the graph). We must note that, even if the condition for the generating set is a generalization of the one imposed by Mans and Shparlinsky, the number of generating sets satisfying it is asymptotically small. Thus the problem of providing a gossiping for circulant graphs is still open.

The results we have found for circulant graphs can be extended to more general Cayley graphs. Let p be a prime number, $d \geq 1$, and $S \subseteq \mathbb{Z}_p^d$. We investigate gossiping in $\Gamma(\mathbb{Z}_p^d, S)$, and note that circulant graphs are the particular case where $d = 1$. We can give an upper bound of the bisection width of these graphs, and therefore bound their gossip complexity. We do so in Theorem 4.4. In Theorem 4.5, we show

Algorithm 2 Gossip($C(n,S)$)

for $i = 1$ to $r-1$ **do**
  for all $\alpha \in \prod_{j=1}^{i-1} \mathbb{Z}_{q_j}$ and $\beta \in \prod_{l=i+1}^{r-1} \mathbb{Z}_{q_l}$, **do in parallel**
    Gossip in $L_{\alpha,\beta} = \{f(\alpha m \beta), m \in \{0,\dots,q_i - 1\}\}$, where $f(a_1 \dots a_{r-1}) = \sum_{i=1}^{r-1} a_i \cdot s_i$
  end do in parallel
end for
procedure GOSSIP $L_{\alpha,\beta}$
  do in parallel
    Gossip $\left\{f(\alpha m \beta), m \in \left\{0,\dots,\left\lfloor \frac{q_i}{2} \right\rfloor - 1\right\}\right\}$ **and**
    Gossip $\left\{f(\alpha m \beta), m \in \left\{\left\lfloor \frac{q_i}{2} \right\rfloor,\dots,r-1\right\}\right\}$
  end do in parallel
  for $l = 0$ to $\left\lfloor \frac{q_i}{2} \right\rfloor - 1$ **do in parallel**
    exchange information between $f(\alpha l \beta)$ and $f(\alpha(q_i - 1 - l)\beta)$
through the path $g(\alpha l \beta, \alpha(q_i - 1 - l)\beta)$, with g defined in the proof of Theorem 4.3.
  end do in parallel
end procedure

that this lower bound is tight (up to a $\log(\log(n))$ factor, where n is the size of the graph).

Theorem 4.4. *Let $S = \{\vec{u_1},\dots,\vec{u_r}\} \subseteq \mathbb{Z}_p^d$. For all $i \in [r]$, we write $\vec{u_i} = (u_1^i \dots, u_d^i)$. For all $l \in [d]$, we write $M_l = \max_{j\in[r]} u_l^j$, and $S_l = \sum_{j=1}^r u_l^j$. Then*

$$g_{\mathrm{VDP}}\left(\Gamma(\mathbb{Z}_p^d, S)\right) \;\geq\; (d+1)\log(p) - \log \min_{l\in[d]} M_l$$

and

$$g_{\mathrm{EDP}}\left(\Gamma(\mathbb{Z}_p^d, S)\right) \;\geq\; (d+1)\log(p) - \log \min_{l\in[d]} S_l.$$

Proof. The idea is the same as in Theorem 4.2. Let $l \in [d]$. We take $V_1 = \mathbb{Z}_p^{l-1} \times \{0,\dots,\lfloor \frac{p}{2} \rfloor - 1\} \times \mathbb{Z}_p^{d-l}$ and $V_2 = \mathbb{Z}_p^l \times \{\lfloor \frac{p}{2} \rfloor,\dots,p-1\} \times \mathbb{Z}_p^{d-l-1}$. Then

$$\partial_{\mathrm{in}}^G(V_1) \;\subseteq\; \mathbb{Z}_p^{l-1} \times \{0,\dots,M_l - 1\} \cup \left\{\lfloor \tfrac{p}{2} \rfloor - M_l,\dots,\lfloor \tfrac{p}{2} \rfloor - 1\right\} \times \mathbb{Z}_p^{d-l},$$

because

$$v_l - u_l \;>\; M_l \bmod p$$

for all $u = (u_1,\dots,u_d) \in \mathbb{Z}_p^{l-1} \times \{M_l,\dots,\lfloor p/2 \rfloor - M_l - 1\} \times \mathbb{Z}_p^{d-l}$ and all $v = (v_1,\dots,v_d) \in V_2$, so that $\{u,v\} \notin E$. Thus, $|\partial_{\mathrm{in}}^G(V_1)| \leq 2M_l p^{d-1}$. This is true for any $l \in [d]$, so $\mathrm{vbw}(G) \leq 2\min_{l\in[d]} M_l p^{d-1}$. So applying Theorem 2.2, we get the result of Theorem 4.4. Similarly, we can show that $|\mathrm{e}(V_1)| \leq 2S_l$ for any $l \in [d]$ and then get the result of Theorem 4.4 for the EDP mode. □

Consider the Cayley graph $\Gamma(\mathbb{Z}_p^d, S)$ with generating set $S = \{\pm s_1,\dots,\pm s_r\}$. For this graph to be connected, we need to have d linearly independent vectors in the set $\{s_1,\dots,s_r\}$, thus in particular $r \geq d$. Moreover, if $r = d$ then $\Gamma(\mathbb{Z}_p^d, S)$ admits the grid $Gr(p,d)$ as a spanning subgraph, and applying the algorithm of [8] gives an optimal gossiping algorithm. So the interesting case is when $r > d$.

Theorem 4.5. *Let p be a prime, and let $d, r \in \mathbb{N}^*$ such that $r > d$. Let $S = \{\pm\vec{u_1}, \dots, \pm\vec{u_r}\} \subseteq \mathbb{Z}_p^d$ such that S generates $\mathbb{Z}_p^d$. For all $i \in [r]$, we write $\vec{u_i} = (u_1^i, \dots, u_d^i)$, and assume that $\vec{u_i} = \lambda_i e_i$ for all $i \in [d-1]$, where e_i is the i-th standard vector as above and $\lambda_i \in \mathbb{Z}_p$.*

If $u_d^d = 1$, $u_d^d < u_d^{d+1} < \cdots < u_d^r$, and $\left\lceil \frac{u_d^{i+1}}{u_d^i} \right\rceil \leq 2\frac{p}{u_d^r}$ for all $i \in \{d, \dots, r-1\}$, then

$$\begin{aligned} g_{\mathrm{VDP}}(\Gamma(\mathbb{Z}_p^d, S)) &\leq (d+2)\log(p) + \log(u_d^r) + 2r - \log(\log(p)) + O(1), \\ g_{\mathrm{VDP}}(\Gamma(\mathbb{Z}_p^d, S)) &\geq 2\log(p) - \log(u_d^r) - \log(\log(u_d^r)) + O(1), \end{aligned}$$

and

$$\begin{aligned} g_{\mathrm{EDP}}(\Gamma(\mathbb{Z}_p^d, S)) &\leq (d+2)\log(p) + \log(u_d^r) + 2r - \log(\log(p)) + O(1), \\ g_{\mathrm{EDP}}(\Gamma(\mathbb{Z}_p^d, S)) &\geq 2\log(p) - \log(ru_d^r) - \log(\log(ru_d^r)) + O(1). \end{aligned}$$

We omit the proof of this theorem, which can be found in [6].

5. Conclusion and open problems

We have given an (almost) optimal gossiping algorithm for a class of circulant graphs. Furthermore, we have shown that we can extend this algorithm to a more general class of Cayley graphs. Finding an optimal gossiping algorithm for all circulant graphs remains an open problem.

It would also be interesting to look for a general algorithm that performs gossiping for a larger class of Cayley graphs. This may also involve looking for better lower bounds for Cayley graphs.

Furthermore, there are other graphs whose structure is close to the hypercube for which we don't know any optimal gossiping algorithm. This the case for the De Bruijn graph. In [6], a gossiping algorithm for this graph is given, but it still far from the known lower bound. In general, any graphs that are good expanders are worth studying for the gossiping problem.

Another interesting problem would be to investigate different kinds of gossiping algorithms. For instance, random gossiping algorithms have been studied for the complete graph [5], for the hypercube [3], or for the grid, but to the best of our knowledge, no such results are known for circulant or Cayley graphs.

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